S3 DMS MAT 203 (CS, IT)

MODULE-3 RELATIONS & FUNCTIONS

Syllabus.

* Cartesian Product, Binary Relation, Function, Domain, Range, One to One Function Image - Restrictions.
* Properties, Reachability Relations, Reflexive, symmetric transitive relations, Antisymmetric Relations.
* Partial order Relation
* Equivalence Relation, Irreflexive Relations
* Partially ordered set (POSET), Hasse Diagram.
* Maximal - Minimal Element, Least upper bound (LUB), Greatest lower bound (GLB)
* Equivalence Relation & partitions, equivalence class.
* Lattice - Dual lattice, Sublattice, properties of glb & lub.
* Properties of lattice, special lattice, Complete lattice, Bounded lattice, Complimented lattice, Distributive lattice.

I

Cartesian Product (Cross Product)

For sets A, B the cartesian product or cross product of A & B is denoted by $A \times B$ and

$A \times B = \{(a,b) \mid a \in A, b \in B\}$

* If A and B are finite $\boxed{|A \times B| = |A| \times |B|}$
* We say that the elements of $A \times B$ are <u>ordered pairs</u>.

<u>Extension of Cartesian product to more than 2 sets.</u>

For sets $A_1, A_2, A_3 \cdots A_n$ the cartesian product

$A_1 \times A_2 \times A_3 \times \cdots \times A_n = \{(a_1, a_2, a_3 \cdots a_n) \mid a_i \in A_i, \forall i\}$

* $\boxed{|A_1 \times A_2 \times A_3 \times \cdots \times A_n| = |A_1| \times |A_2| \times |A_3| \times \cdots \times |A_n|}$
* We say that the elements of $A_1 \times A_2 \times \cdots \times A_n$ are called <u>ordered n-tuples</u>

A <u>3-tuple</u> is also known as a <u>triple</u>.

1) Let $A = \{1,2,3\}$ $B = \{x, y\}$ find $A \times B$, $B \times A$, B^2, B^3

$A \times B = \{(1,x), (1,y), (2,x), (2,y), (3,x) (3,y)\}$

$B \times A = \{(x,1), (x,2), (x,3), (y,1) (y,2), (y,3)\}$

$B^2 = B \times B = \{(x,x), (x,y), (y,x), (y,y)\}$

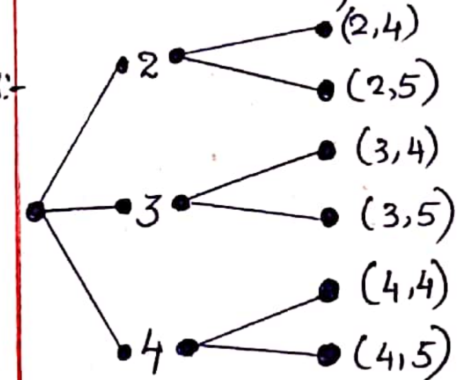
$B^3 = B \times B \times B = \{(x,x,x), (x,x,y), (x,y,x), (x,y,y), (y,x,x), (y,x,y)$
$(y,y,x), (y,y,y)\}$

TREE DIAGRAM:- The pictorial representation of A×B

Eg:- $A = \{2,3,4\}$ $B = \{4,5\}$ $C = \{x,y\}$

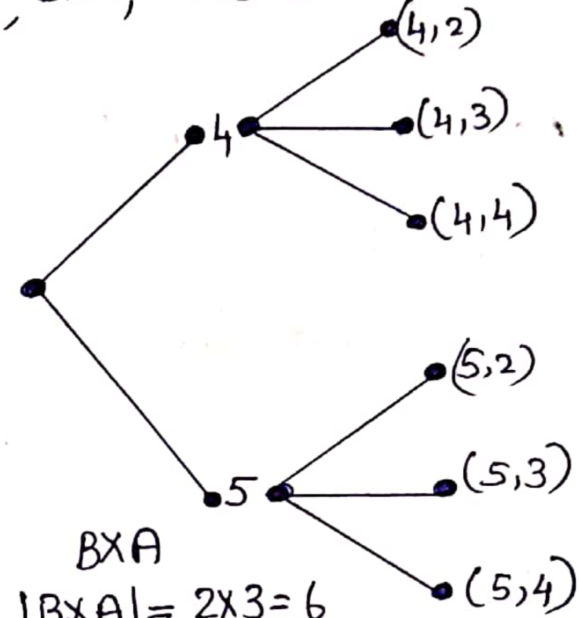
Draw tree diagrams of A×B, B×A, A×B×C and find $|A\times B|$, $|B\times A|$, $|A\times B\times C|$

Ans:-



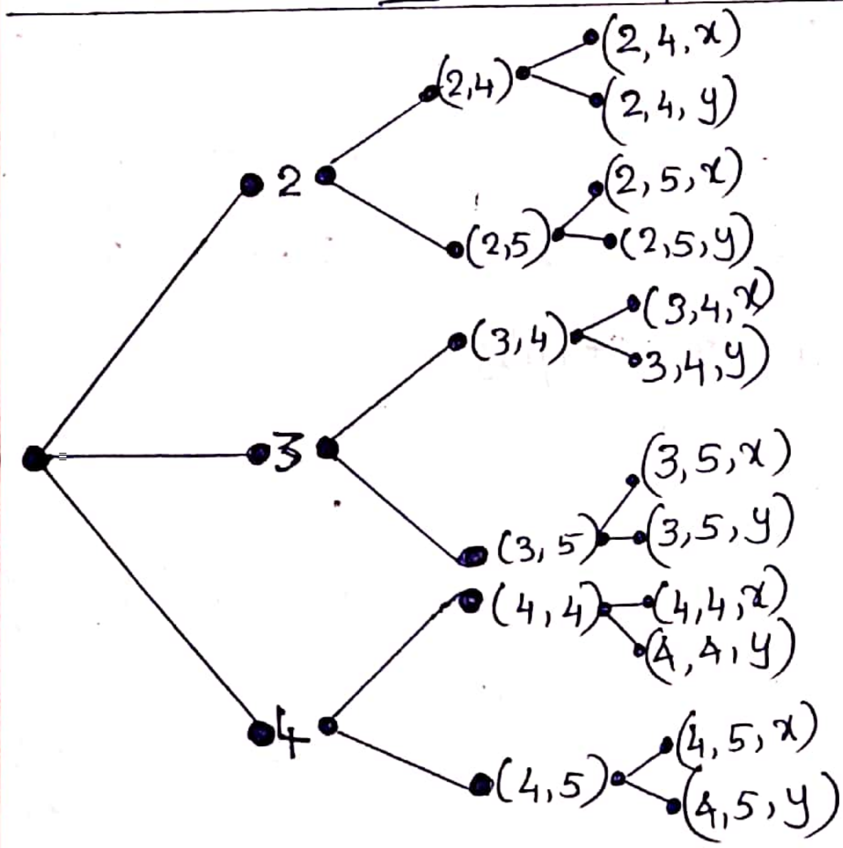
A×B

$|A\times B| = 3\times 2 = \underline{\underline{6}}$



B×A

$|B\times A| = 2\times 3 = \underline{\underline{6}}$



A×B×C.

$|A\times B\times C| = 3\times 2\times 2$
$= \underline{\underline{12}}$

BINARY RELATION (RELATION)

For sets A,B any subset of $A \times B$ is called a binary (Relation) from A to B.
Any subset of $A \times A$ is called a relation on A.

<u>Note</u> :- Since we deal with binary relations, the word "relation" will mean binary relation, unless something otherwise is specified.

Total number of relations from (A to B) & (B to A)

If $|A| = m$, $|B| = n$, then total number of relations $= \boxed{2^{mn}}$ (for both $A \to B$ & $B \to A$) ($\because 2^{mn} = 2^{nm}$)

Notation $\mathcal{P}$

If $A = \{1, 2, 3\}$ then $\mathcal{P}(A)$ denotes the set of all subsets of A

ie $\mathcal{P}(A) = \{\phi, \{1\}, \{2\}, \{3\}, \{1,2\}, \{1,3\}, \{2,3\}, \{1,2,3\}\}$ are the possible subsets. ($2^3 = 8$ subsets).

$$\boxed{\text{If } |A| = n \text{ then } |\mathcal{P}(A)| = 2^n.}$$

RESULT:-

For any sets $A, B, C \subseteq \mathcal{U}$

a) $A \times (B \cap C) = (A \times B) \cap (A \times C)$

b) $A \times (B \cup C) = (A \times B) \cup (A \times C)$

c) $(A \cap B) \times C = (A \times C) \cap (B \times C)$

d) $(A \cup B) \times C = (A \times C) \cup (B \times C)$

PROBLEMS.

1) Let $A = \{a, b, c, d\}$ $B = \{1, 2, 3\}$

a) Give examples ~~for~~ of 3 non empty relations from A to B

b) 3 examples of nonempty relations on A.

Ans (a) (i) Relation is a subset of $A \times B$..

$\{(a,1), (a,2), (a,3)\}$

(ii) $\{(b,2), (b,3), (c,2), (d,3)\}$

(iii) $\{(d,3), (c,1), (b,3), (d,1), (a,1), (a,2)\}$. etc

(b) (i) $\{(a,b), (a,c), (a,d)\}$

(ii) $\{(a,a)\}$

(iii) $\{(d,a), (d,c), (d,b), (c,b), (b,c), (a,d), (d,a)\}$

2) Let $A = \{a, b, c, d, e, f, g, h\}$, $B = \{1, 2, 3, 4, 5\}$

How many elements are there in $P(A \times B)$.

Ans:- $A \times B = \{(a,1), (a,2), \ldots (h,5)\}$

$|A \times B| = |A| \times |B| = 8 \times 5 = 40$

$\therefore |P(A \times B)| = \underline{\underline{2^{40}}}$ is the number of subsets of $A \times B$. $(P(A \times B))$

3) If $A = \{1, 2, 3, 4\}$
H.W $B = \{2, 5\}$
$C = \{3, 4, 7\}$. Determine

(i) $A \times B$
(ii) $B \times A$
(iii) $A \cup (B \times C)$
(iv) $(A \cup B) \times C$
(v) $(A \times C) \cup (B \times C)$

2) Let $A = \{a, b, c, d, e, f, g, h\}$, $B = \{1, 2, 3, 4, 5\}$

How many elements are there in $P(A \times B)$.

Ans:- $A \times B = \{(a,1), (a,2), \ldots \ldots (h,5)\}$.

$|A \times B| = |A| \times |B| = 8 \times 5 = 40$

$\therefore |P(A \times B)| = \underline{\underline{2^{40}}}$ is the number of subsets of $A \times B$. ($P(A \times B)$

3) If $A = \{1, 2, 3, 4\}$

H.W $B = \{2, 5\}$

$C = \{3, 4, 7\}$. Determine

(i) $A \times B$

(ii) $B \times A$

(iii) $A \cup (B \times C)$ Don't use the formula.

(iv) $(A \cup B) \times C$

(v) $(A \times C) \cup (B \times C)$

Ans (i) $A \times B = \{(1,2) (1,5), (2,2), (2,5), (3,2), (3,5), (4,2), (4,5)\}$

(ii) $B \times A = \{(2,1), (2,2), (2,3), (2,4), (5,1), (5,2), (5,3), (5,4)\}$

(iii) $A \cup (B \times C) = \{1, 2, 3, 4, (2,3), (2,4), (2,7), (5,3), (5,4), (5,7)\}$

(iv) $A \cup B = \{1, 2, 3, 4, 5\}$ $C = \{3, 4, 7\}$

$(A \cup B) \times C = \{(1,3), (1,4), (1,7), (2,3), (2,4), (2,7), (3,3), (3,4), (3,7), (4,3), (4,4), (4,7), (5,3), (5,4), (5,7)\}$

(v) $A \times C = \{(1,3), (1,4), (1,7), (2,3), (2,4), (2,7), (3,3), (3,4), (3,7), (4,3), (4,4), (4,7)\}$

$B \times C = \{(2,3), (2,4), (2,7), (5,3), (5,4), (5,7)\}$

$(A \times C) \cup (B \times C) = \{(1,3), (1,4), (1,7), (2,3), (2,4), (2,7), (3,3), (3,4), (3,7), (4,3), (4,4), (4,7), (5,3), (5,4), (5,7)\}$

4) If $A = \{1,2,3\}$ $B = \{2,4,5\}$ give examples of

H.W

(a) 3 nonempty relations from A to B.

(b) 3 " " on A

(c) $|A \times B|$

(d) The number of relations from A to B

(e) The number of relations on A

A

(a) (i) $\{(1,2),(1,4)\}$

(ii) $\{(2,4),(2,2),(2,5),(3,5)\}$

(iii) $\{(3,2),(3,4),(3,5),(1,2),(1,4)\}$

give any examples

(b) (i) $\{(1,1)\}$

(ii) $\{(1,1),(1,2),(1,3),(3,1)\}$

(iii) $\{(2,1),(2,2),(3,1),(3,2),(3,3)\}$

A: $\{1,2,3\}$

A: $\{1,2,3\}$

$A \times A$

(c) $|A \times B| = |A| \cdot |B| = 3 \times 3 = 9$

(d) No of relations from A to B $= 2^{|A||B|} = 2^{3\times 3} = \underline{\underline{2^9}}$

(e) No of relations ~~from~~ on A $(A \to A) = 2^{3\times 3} = \underline{\underline{2^9}}$

$2^{|A||A|} =$

5) If $A = \{1,2,3,4,5\}$ and $B = \{w, x, y, z\}$. How many elements are there in $P(A \times B)$

Ans:- $|P(A\times B)| = 2^{|A\times B|} = 2^{5\times 4}$ $\quad |A\times B| = |A|\,|B| = 5\times 4 = 20$

$$= \underline{\underline{2^{20}}}$$

6) H.W Let A, B be sets with $|B| = 3$. If there are 4096 relations from A to B what is $|A|$?

Ans:- Let $R: A \rightarrow B$

no of relations $= 2^{mn}$ where $|A| = m$ $|B| = n$.

$4096 = 2^{m3}$ given $|B| = 3 = n$.

$$\begin{array}{r|l} 2 & 4096 \\ 2 & 2048 \\ 2 & 1024 \\ 2 & 512 \\ 2 & 256 \\ 2 & 128 \\ 2 & 64 \\ 2 & 32 \\ 2 & 16 \\ 2 & 8 \\ 2 & 4 \\ & 2 \end{array}$$

find m.

$4096 = 2^{12}$

$4096 = 2^{12} = 2^{3m}$

$12 = 3m \quad \therefore m = \frac{12}{3} = \underline{\underline{4}}$

$\therefore |A| = \underline{\underline{4}}$

II. FUNCTIONS (5.2)

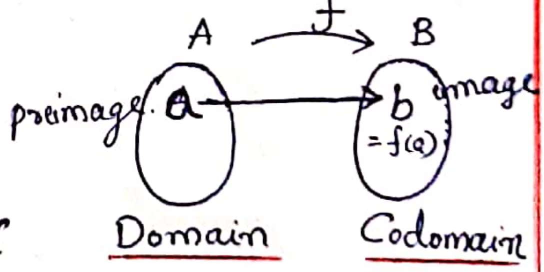
For non empty sets A, B a function or mapping 'f' from A to B denoted by $f: A \to B$ is a relation from A to B in which every element of A appears exactly once as the first component of an ordered pair in the relation.

In other words, a function $f: A \to B$ is an assignment of exactly one element of B to every element of A.

Note (Image, preimage, Domain, Codomain Range)

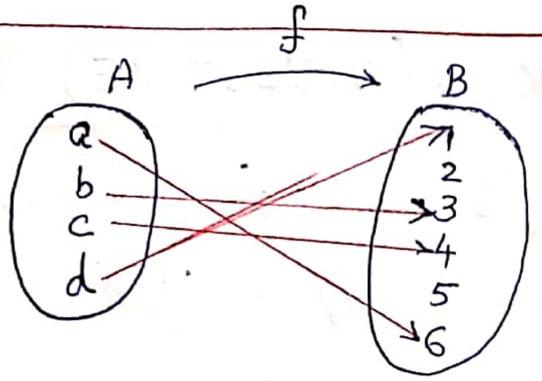
* $f(a) = b$ when (a, b) is an ordered pair in f.
  b is called the image of a under f
  a is the preimage of b under f

* Let $f: A \to B$
  A is called the domain of f
  B is called the co-domain of f



* The set of all images of A is called Range.

(1)



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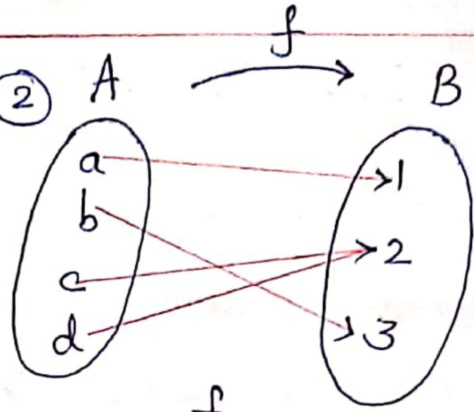
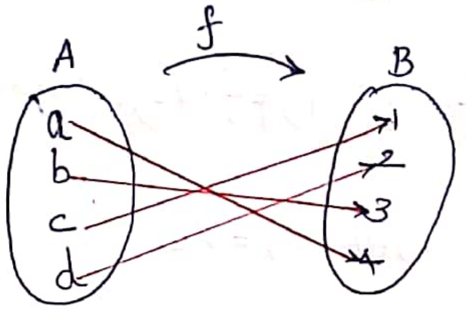
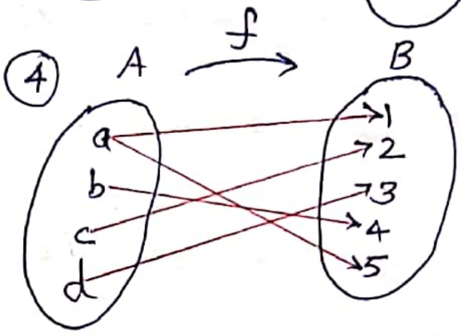


(3)



(4)

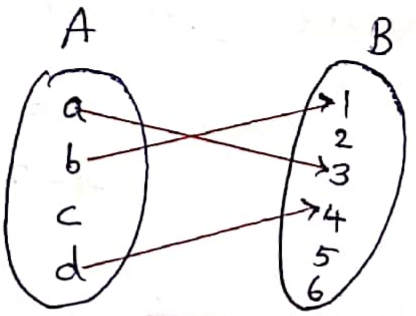


Not a function, because a has 2 images

i.e. $f(a)=1$, $f(a)=5$

Not possible

(5)



Not a function, because c has no image

Let $f: A \to B$ be a function. with $|A| = m$, $|B| = n$

Total $\boxed{\text{number of functions from } A \to B \text{ is } |B|^{|A|}} = n^m$

ONE TO ONE FUNCTIONS (INJECTIVE)

$f: A \to B$ is called one to one or injective if distinct elements of A are mapped into distinct elements of B.

ie f is one to one iff $f(x_1) \neq f(x_2)$ whenever $x_1 \neq x_2$

OR

$f(x_1) = f(x_2)$ when ever $x_1 = x_2$.

The number of one to one functions from A to B is $^nP_m = {}^{|B|}P_{|A|} = \dfrac{n!}{(n-m)!}$ where $|A| = m$, $|B| = n$

Note:- $f: A \to B$ is one to one, then $|A| \leq |B|$

In page no: 9 Egs ① & ③ are one - one functions.

1) Check whether the following function $f: R \to R$ where $f(x) = 3x+7$ $\forall x \in R$ is one to one.

Ans: Let $f(x_1) = f(x_2)$

$3x_1 + 7 = 3x_2 + 7$

$3x_1 = 3x_2$

$x_1 = x_2$

$\therefore f$ is one to one.

② $g:- R \to R$ $g(x) = x^4 - x$

Ans:- $g(x_1) = g(x_2)$

$x_1^4 - x_1 = x_2^4 - x$

Let $g(0) = 0$, $g(1) = 1$

$0^2 - 0 = 0$, $1^2 - 1 = 0$

but $0 \neq 1$

ie g is not one to one

ie $\exists\, x_1, x_2$ where $g(x_1) = g(x_2)$ but $x_1 \neq x_2$

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The number of one to one functions from A to B is $^nP_m = {}^{|B|}P_{|A|} = \dfrac{n!}{(n-m)!}$ where $|A| = m$, $|B| = n$

Note:- $f: A \to B$ is one to one, then $|A| \leq |B|$

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1) Check whether the following function $f: R \to R$ where $f(x) = 3x+7$ $\forall x \in R$ is one to one.

Ans: Let $f(x_1) = f(x_2)$

$3x_1 + 7 = 3x_2 + 7$

$3x_1 = 3x_2$

$x_1 = x_2$

$\therefore f$ is one to one.

② $g: R \to R$ $g(x) = x^4 - x$

Ans:- $g(x_1) = g(x_2)$

$x_1^4 - x_1 = x_2^4 - x$

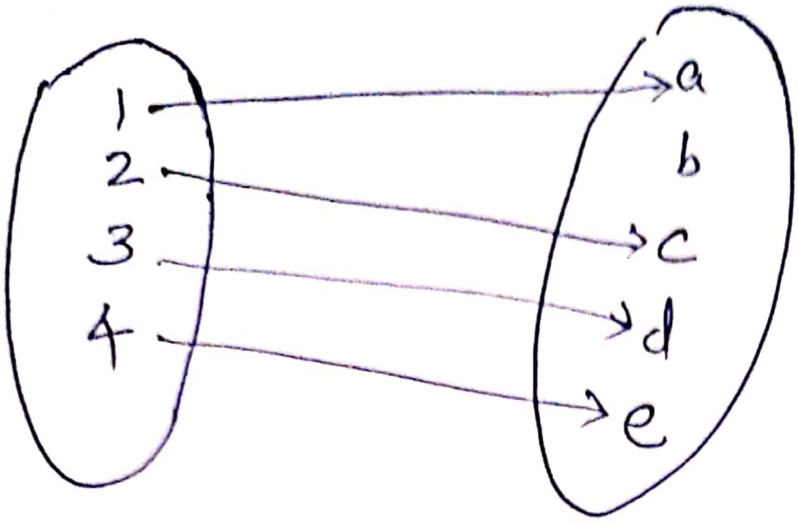
Let $g(0) = 0$, $g(1) = 1$

$0^2 - 0 = 0$ $\quad 1^2 - 1 = 0$

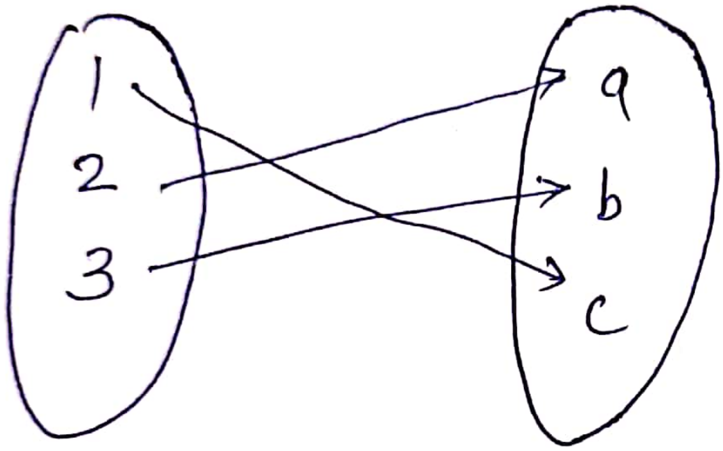
but $0 \neq 1$

ie g is not one to one

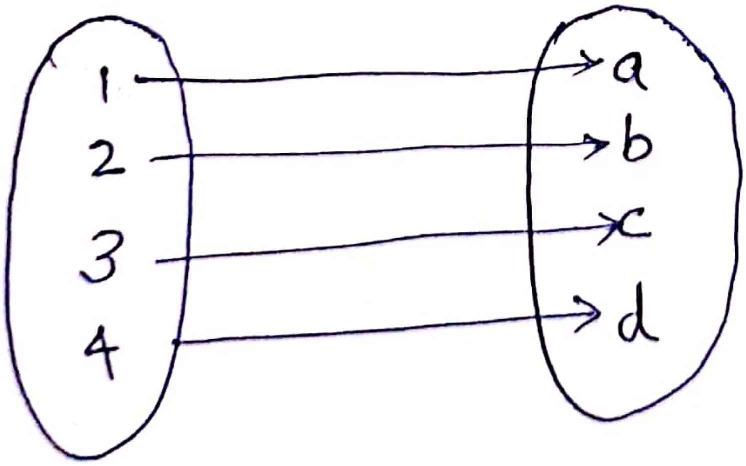
ie $\exists\, x_1, x_2$ where $g(x_1) = g(x_2)$ but $x_1 \neq x_2$



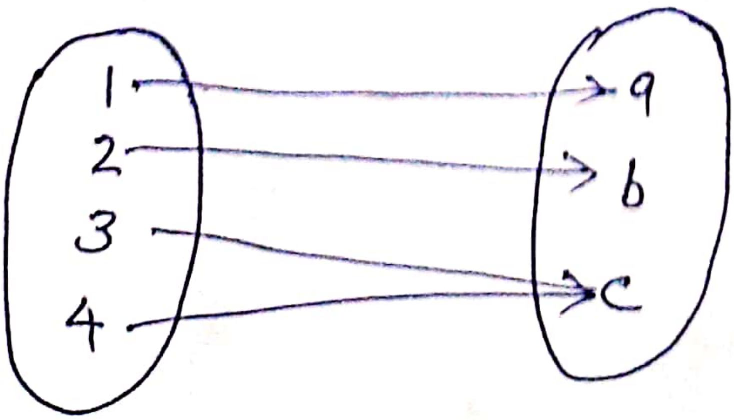
one one



one one



one one

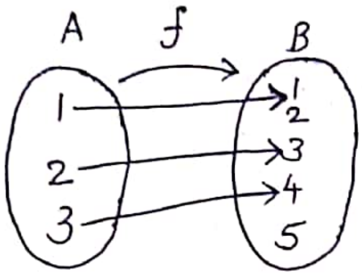


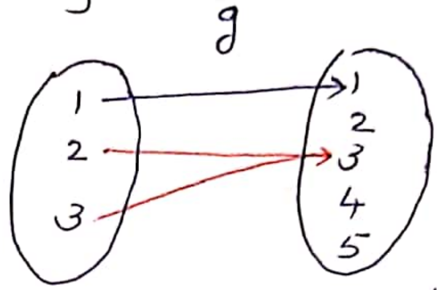
not one one

3\. Let $A = \{1, 2, 3\}$ and $B = \{1, 2, 3, 4, 5\}$ Check whether the functions $f : \{(1,1), (2,3), (3,4)\}$

$g : \{(1,1), (2,3), (3,3)\}$ are one to one

Ans:-





f is one to one

Every element in A has different images in B

g is not one to one

because $g(2) = g(3)$ but $2 \neq 3$.

2 & 3 have same image 3 in B.

$\therefore$ not one to one

4\. How many functions are there from $f : A \to B$ & how many of them are one to one.

(A & B are given above in Q.3)

Ans:- Total number of functions $= |B|^{|A|} = \underline{\underline{5^3}} = \underline{\underline{125}}$

Number of one to one functions $= {}^{|B|}P_{|A|} = {}^5P_3$

$$= \frac{5!}{(5-3)!} = \frac{5!}{2!}$$

$$= \frac{5 \times 4 \times 3 \times 2 \times 1}{1 \cdot 2}$$

$$= \underline{\underline{60}}$$

~~Definition~~

5) Determine which of the following functions are one to one and find its range.

(a) $f: Z \to Z : f(x) = 2x$

(b) $f: Q \to Q : f(x) = 2x$

(c) $f: R \to R : f(x) = e^{x^2}$

(d) $f: \left[\frac{-\pi}{2}\ \frac{\pi}{2}\right] \to R : f(x) = \cos x$

H.W

(e) $f: Z \to Z,\ f(x) = 2x+1$

(f) $f: Q \to Q,\ f(x) = 2x+1$

(g) $f: Z \to Z,\ f(x) = x^3 - x$

(h) $f: R \to R,\ f(x) = e^x$

(i) $f: \left[\frac{-\pi}{2}, \frac{\pi}{2}\right] \to R \quad f(x) = \sin x$

(j) $f: [0, \pi] \to R \quad f(x) = \sin x$

Std NOTATIONS

Z : set of integers $\{\ldots -4, -3, -2, -1, 0, 1, 2, \ldots\}$

Z^+ : $\{1, 2, 3, 4, \ldots\}$ +ve integers

Z^- : $\{\ldots -4, -3, -2, -1\}$ –ve integers

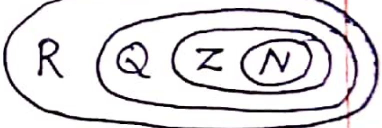
Q : Set of Rational numbers

fractions $\{\frac{4}{7}, 1, \frac{4}{3}, 12\frac{7}{16}, 202, \frac{31}{129} \ldots\}$

$\{\frac{p}{q} / p, q \in Z,\ q \neq 0\}$

R : Set of all real numbers $\{-1, -\frac{1}{2}, \sqrt{2}, \pi, 9 \ldots\}$

N : Set of Natural numbers $\{1, 2, 3, 4, \ldots\}$ Counting numbers



Ans (a) $f(x_1) = f(x_2)$

$2x_1 = 2x_2$

$x_1 = x_2$

ie $f(x_1) = f(x_2) \Rightarrow x_1 = x_2$

$\therefore$ (a) is a one to one function.

Range $= \{2x / x \in Z\}$ means set of even numbers.

$= \{2, 4, 6, 8, \ldots\}$

(b) is a one to one function.

Range $= \{2x / x \in Q\} = Q$.

(c) $x_1 = -1, \quad x_2 = +1$

$f(x_1) = e^{x_1^2} = e^{(-1)^2} = e, \quad f(x_2) = e^{x_2^2} = e^{1^2} = e$

ie $x_1 \neq x_2 \Rightarrow f(x_1) = f(x_2)$

$\therefore$ not one to one.

Range $= \{[0, \infty)\}$

(d) $x_1 = -\frac{\pi}{2}$ $\qquad$ $x_2 = \frac{\pi}{2}$

$f(x_1) = \cos(-\frac{\pi}{2})$ $\qquad$ $f(x_2) = \cos\frac{\pi}{2}$

$= \cos\frac{\pi}{2} = 0$ $\qquad$ $\cos\frac{\pi}{2} = 0$

ie $x_1 \neq x_2 \Longrightarrow f(x_1) = f(x_2)$ $\quad \therefore$ not one to one

Range = $\underline{\underline{[0, 1]}}$

Definition <u>Image of a subset of A</u>

If $f: A \to B$ and $A_1 \subseteq A$ then

$f(A_1) = \{ b \in B / f(a) = b$ for some $a \in A_1 \}$ and

$f(A_1)$ is called image of A_1 under f.

1) $A = \{1, 2, 3, 4, 5\}$

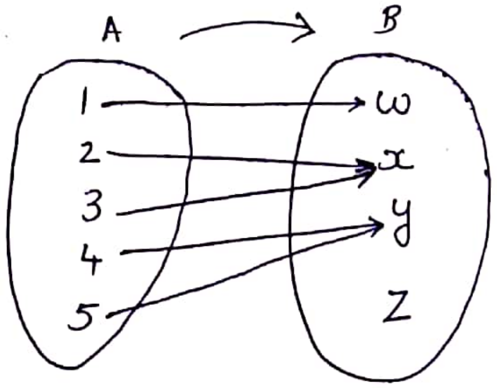
$B = \{w, x, y, z\}$ Let $f: A \to B$ be given by

$f = \{ (1, w), (2, x), (3, x), (4, y), (5, y) \}$. Let $A_1 = \{1\}$,

$A_2 = \{1, 2\}$, $A_3 = \{1, 2, 3\}$, $A_4 = \{2, 3\}$, $A_5 = \{2, 3, 4, 5\}$

Find $f(A_1)$, $f(A_2)$, $f(A_3)$, $f(A_4)$, $f(A_5)$ | Images set of A_1, A_2, A_3, A_4, A_5

Ans:-



$f(A_1) = \{ f(1) \} = \{ w \}$

$f(A_2) = \{ f(1), f(2) \} = \{ w, x \}$

$f(A_3) = \{ f(1), f(2), f(3) \} = \{ w, x \}$

$f(A_4) = \{ x \}$

$f(A_5) = \{ x, y \}$

2) $g: R \to R$ given by $g(x) = x^2$. Let $A_1 = [-2, 1]$ (closed interval)

What is the image of A_1.

Ans:- $A_1 = [-2, 1] \subset R$

$A_1 = \{ -2, \cdots -1, \cdots 0, \cdots \frac{1}{2} \cdots 1 \}$

$g(A_1) = \{ g(-2) \cdots, g(-1), \cdots g(0), \cdots g(1) \} = \{ 4, \cdots 1, \cdots 0, \cdots \frac{1}{4}, \cdots 1 \}$

$\therefore g(A_1) = [0, 4]$ closed interval.

H.W
3) Let $f: R \to R$ where $f(x) = x^2$. Determine $f(A)$ for the following.

a) $A = \{2, 3\}$ b) $A = (-3, 3)$ c) $A = [-7, 2]$

d) $A = (-3, 2]$ f) $[-4, -3] \cup [5, 6]$

4) If $A = \{1, 2, 3, 4, 5\}$ and there are 6720 injective (one-to one) functions from $f: A \to B$ what is $|B|$

H.W

Ans:- No of one to one functions = ${}^{|B|}P_{|A|}$ =

Hint

$|A| = 5$

$|B| = n$

ie ${}^{n}P_5 = 6720$

$$\frac{n!}{(n-5)!} = 6720$$

H.W
5) If there are 2187 functions $f: A \to B$ and $|B| = 3$, what is $|A| = ?$

Hint Total number of functions from $A \to B$ is $|B|^{|A|}$

$2187 = 3^m$

Let $|A| = m$
find m.

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RESULT

Let $f: A \to B$ with $A_1, A_2 \subseteq A$ then

ⓐ $f[(A_1) \cup (A_2)] = f(A_1) \cup f(A_2)$

ⓑ $f[(A_1) \cap (A_2)] \subseteq f(A_1) \cap f(A_2)$

ⓒ $f(A_1 \cap A_2) = f(A_1) \cap f(A_2)$ when f is one-one.

RESTRICTION of a function f to A_1 (f/A_1)

If $f: A \to B$ where $A_1 \subseteq A$ then

$f/A_1 : A_1 \to B$ is called the restriction of f to A_1

if $f/A_1(a) = f(a)$ for all $a \in A_1$

$\boxed{\text{choosing a smaller domain } A_1 \text{ for the original function } f}$

EXTENSION OF f to A

Let $A_1 \subseteq A$ and $f: A_1 \to B$.

If $g: A \to B$ and $g(a) = f(a)$ $\forall\, a \in A_1$, then

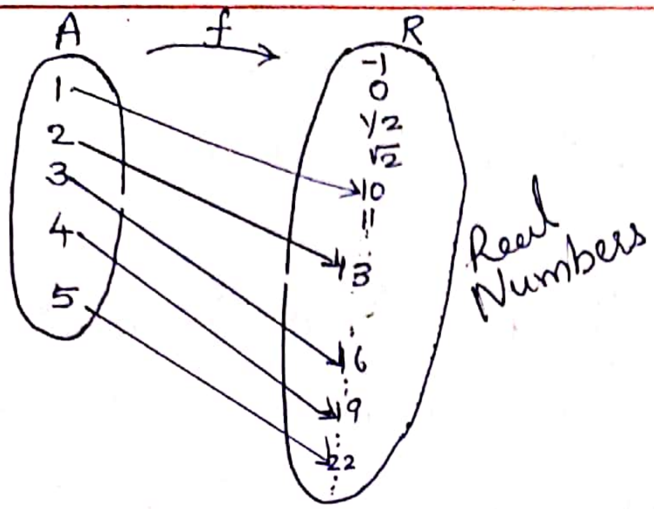
we call g an extension of f to A

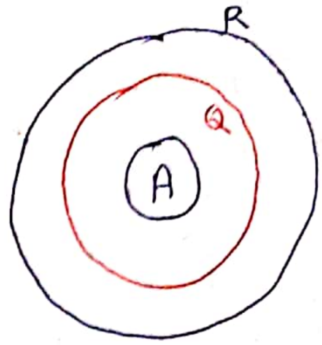
Eg.- For $A = \{1, 2, 3, 4, 5\}$ $\boxed{f: A \to R}$ defined by

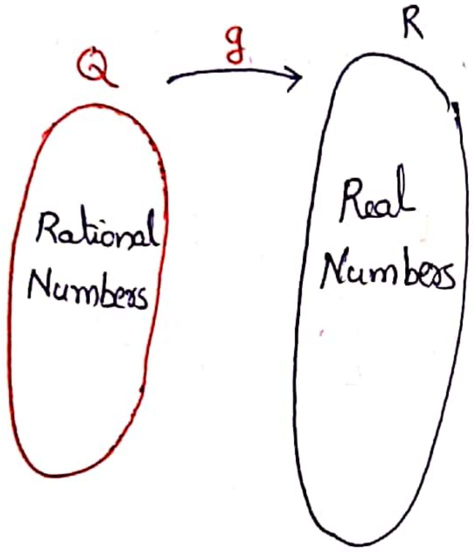
$f = \{(1, 10), (2, 13), (3, 16), (4, 19), (5, 22)\}$

Let $\boxed{g: Q \to R}$ defined by $g(q) = 3q + 7$ $\forall\, q \in Q$.

finally $\boxed{h: R \to R}$ defined by $h(r) = 3r + 7$ $\forall\, r \in R$.







(i) f is the restrn of g (from Q) to A

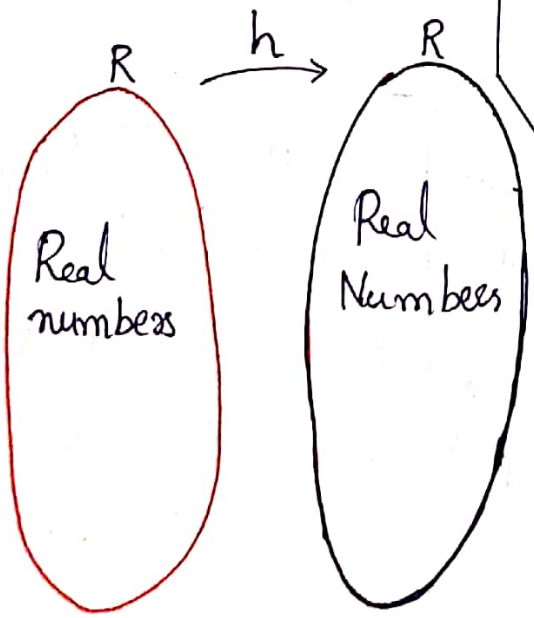
(ii) f is the restrn of h (from R) to A

(iii) h is an extension of f (from A) to R

(iv) h is an extension of g (from Q) to R

(v) g is an extension of f (from A) to Q.

(vi) g is a restriction of h (from R) to Q



Q) Determine whether or not each of the following relations are functions. If a relation is a function find its range

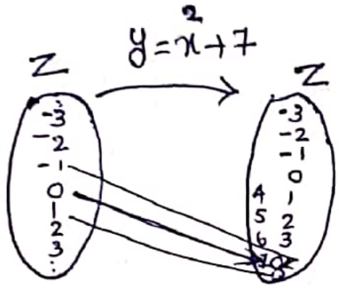
(i) $\{(x,y) / x,y \in Z,\ y = x^2+7\}$ a relation from Z to Z

(ii) $\{(x,y) / x,y \in R,\ y^2 = x\}$, a relation from R to R

H.W (iii) $\{(x,y) / x,y \in R,\ y = 3x+1\}$, a relation from R to R

H.W (iv) $\{(x,y) / x,y \in Q,\ x^2+y^2=1\}$ a relation from Q to Q

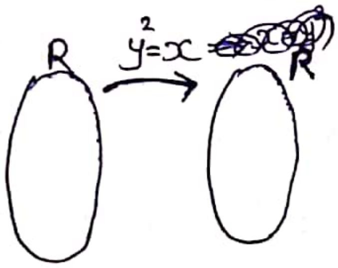
Ans:- (1)



A relation and a function

$\therefore$ Range $= \{7, 8, 11, 16, 23, \cdots\}$

(ii)



$y^2 = x$

$y = \pm\sqrt{x}$

when $x = -3$

we cannot map -3 to a +ve real number.

-ve numbers have no image

$\therefore$ A relation, but not a function.

$\therefore$ No range

(iii)

H.W

Q Check whethere $\left.\begin{matrix} f: R \to R \\ \\ f: Z \to R \end{matrix}\right\}$ where $f(x) = \dfrac{1}{x^2-2}$ and where $f(x) = \dfrac{1}{x^2-2}$ are functions or not

Ans:- $f: R \to R$, $R \xrightarrow{f} R$ when $x = \sqrt{2} \in R$ (A real number)

$$f(x) = \frac{1}{(\sqrt{2})^2 - 2} = \frac{1}{0} \notin R \text{ (has no image)}$$

$\therefore$ $f: R \to R$ <u>is not a function</u> under $f(x) = \dfrac{1}{x^2-2}$

$f: Z \to R$ $\quad Z \xrightarrow{f} R$

$x = \{\dots -3, -2, -1, 0, 1, 2, 3, 4, \dots\} \in Z$

$f(x) = \left\{ \dfrac{1}{x^2-2} \right\}$ a real number

$\therefore$ Every element in Z has exactly one image $\therefore$ <u>A function</u>

Range $= \underline{\{R\}}$

Relations : Properties of Relations:-

1. Reflexive Relations.

A relation R on a set A is called reflexive if $xRx \ \forall x \in A$ ie $(x,x) \in R. \ \forall x \in A$

2. Symmetric Relations.

A relation R on a set A is called symmetric if $xRy \Rightarrow yRx \ \forall \ x,y \in A.$

ie if $(x,y) \Rightarrow (y,x)$

3. Transitive Relations.

A relation R on a set A is called transitive if $xRy,\ yRz \Rightarrow xRz \quad \forall \ x,y,z \in A$

ie if $(x,y),(y,z) \Rightarrow (x,z) \quad \forall \ x,y,z \in A$

here 'y' plays the role of intermediary

Egs of Reflexive Relations	Egs of Symmetric Relation.	Egs of Transitive Relations.
① Relation '$\leq$' on Z ② Relation $\subseteq$ on collection of sets ③ Relation divisibility on N ④ Relation parallel to in the case of lines. ⑤ Relation $=$ on R	① The relation $\perp$r to. ② The relation parallel to ③ Relation $=$	① $\leq$, $\subseteq$, divisible by ② Relation $=$

4) Antisymmetric Relation.

If aRb and bRa then $a=b$, R is called antisymmetric $\forall a,b \in A$.

Eg:- $\subseteq$, $\leq$ are antisymmetric relation

$A \subseteq B$ and $B \subseteq A \Longrightarrow A = B$

$a \leq b$ and $b \leq a \Longrightarrow a = b$.

EQUIVALENCE RELATION

An equivalence relation R on a set A is a relation that is reflexive, symmetric and transitive

PARTIAL ORDER RELATION

A relation R on a set A is called a partial order, or a partial ordering relation, if R is reflexive, antisymmetric and transitive

1) Let $A = \{1, 2, 3\}$. Check whether following functions are reflexive, symmetric or both.

a) $R_1 = \{(1,2), (2,1), (1,3), (3,1)\}$

b) $R_2 = \{(1,1), (2,2), (3,3), (2,3)\}$

c) $R_3 = \{(1,1), (2,2), (3,3)\}$

d) $R_4 = \{(1,1), (2,2), (3,3), (2,3), (3,2)\}$

e) $R_5 = \{(1,1), (2,3), (3,3)\}$

Ans:- a) R_1 is Symmetric because $1R2 \Rightarrow 2R1$ ie $(1,2) \Rightarrow (2,1)$
$1R3 \Rightarrow 3R1$ ie $(1,3) \Rightarrow (3,1)$

but not reflexive because no element is related to it self. $1 \not R 1$, $2 \not R 2$, $3 \not R 3$. $(1,1), (2,2), (3,3)$ are not members of R_1.

b) R_2 is reflexive but not symmetric.
reflexive because $(1,1)\ (2,2), (3,3) \in R_2$
not symmetric because $(2,3) \in R_2$ but $(3,2) \notin R_2$.

c) R_3 is reflexive and symmetric (write reasons)

d) R_4 is reflexive and symmetric (write reasons)

e) R_5 is neither reflexive nor symmetric
because $(2,2), (3,3) \notin R_5$ (not reflexive)
$(2,3) \in R_5$ but $(3,2) \notin R_5$ (not symmetric

2) Let $A=\{1,2,3,4\}$ check whether the following are reflexive, symmetric, transitive or not.

ⓐ $R_1=\{(1,1)(2,3),(3,4),(2,4)\}$

ⓑ $R_2=\{(1,3),(3,2)\}$

Ans: ⓐ R_1 not reflexive $(\because (2,2),(3,3),(4,4) \notin R_1)$

not symmetric $(\because (3,2),(4,3),(4,2) \notin R_1)$

transitive $(\because (2,3),(3,4) \in R_1 \Rightarrow (2,4) \in R_1)$

ⓑ R_2 not reflexive (no element is related itself)

not symmetric $((3,1),(2,3) \notin R_2)$

not transitive $((1,3),(3,2) \in R_2$ but $(1,2) \notin R_2.)$

---

3) Check whether the following relations are equivalence relations.

$A=\{1,2,3\}$

ⓐ $R_1=\{(1,1),(2,2),(3,3)\}$

ⓑ $R_2=\{(1,1),(2,2),(2,3),(3,2),(3,3)\}$

ⓒ $R_3=\{(1,1),(1,3),(2,2),(3,1),(3,3)\}$

ⓓ $R_4=\{(1,1),(1,2),(1,3),(2,1),(2,2),(2,3),(3,1),(3,2),(3,3)\}=A\times A$

Ans:- All are equivalence relations

because

4) If $A = \{1,2,3,4,5\}$ give examples of a relation R on A that

(a) reflexive & symmetric but not transitive.

(b) reflexive & transitive but not symmetric

(c) symmetric & transitive but not reflexive

Ans:- (a) $R_1 = \{\overbrace{(1,1),(2,2),(3,3),(4,4),(5,5)}^{\text{reflexive}}, \overbrace{(1,3),(3,1)}^{\text{symmetric}}, \overbrace{(3,4),(4,3)}^{\text{symmetric}}\}$

not transitive because $(1,3),(3,4) \in R_1$ but $(1,4) \notin R_1$

(b) $R_2 = \{\overbrace{(1,1),(2,2),(3,3),(4,4),(5,5)}^{\text{reflexive}}, \overbrace{(1,3),(3,4),(1,4)}^{\text{Transitive}}\}$

not symmetric because
$(1,3) \in R_2$ but $(3,1) \notin R_2$
$(3,4) \in R_2$ but $(4,3) \notin R_2$
$(1,4) \in R_2$ but $(4,1) \notin R_2$

(c) $R_3 = \{\overbrace{(1,3),(3,1)}^{\text{symmetric}},(1,1)\}$ with $\underbrace{(1,3),(3,1),(1,1)}_{\text{Transitive}}$

Not reflexive because $\{(2,2),(3,3),(4,4),(5,5) \notin R_3\}$

---

5) Check whether R on the set Z^+ defined by aRb if a/b (read it as a (exactly) divides b) ie $b = ma$ for some $m \in Z^+$, is an equivalence relation.

Ans:- <u>reflexive</u>

aRa ie a/a ie $a = 1 \times a$
$\therefore$ <u>R is reflexive</u>.

<u>Symmetric</u>

if aRb ie a/b
ie $b = ma$ then $a = \frac{1}{m} b$ $(b \nmid a)$ — b doesnot divide a

Counter Eg: if $2R6$, but $6 \not R 2$ $\therefore$ <u>not symmetric</u>

Divides - meaning (Egs)

$5/15$ (5 divides 15) $\Rightarrow 15 = m5$
$15 = 3 \times 5$

$7/21$ (7 divides 21) $\Rightarrow 21 = m7$
$21 = 3 \times 7$

$2/48$ (2 divides 28) $\Rightarrow \frac{48}{2} = 12$
$48 = 2 \times 12$

Transitive

If (aRb) and (bRc) (ie a divides b and b divides c)

ie $b = ma$ and

$c = nb$ for some $m, n \in Z^+$

then $c = n(ma)$

$c = (nm)a$

$c = la$ $\quad l \in Z^+ \; (\because m, n \in Z^+)$

ie a divides c aRc

so we have aRb and bRc $\Rightarrow$ aRc $\therefore$ Transitive

R is reflexive & transitive but not symmetric.

$\therefore$ R on Z^+ defined by aRb or a divides b is not an equivalence relation.

6) R on Z defined by aRb when $ab \geq 0$.
Is it an equivalence relation.

Ans:- Reflexive

aRa because $a^2 \geq 0 \quad \forall a \in Z$

$\therefore$ R is reflexive

symmetric

aRb means $ab \geq 0$, which is same as $ba \geq 0$

ie bRa

$\therefore aRb \Rightarrow bRa$

$\therefore$ R is symmetric.

Transitive

Let aRb and bRc $a = -2$

ie $ab \geq 0$ and $bc \geq 0$ $b = 0$

$c = 4$

counter example

$(-2)(0) = 0 \geq 0$ aRb

$(0)(4) = 0 \geq 0$ bRc

then $(-2)(4) = -8 \not\geq 0$ $a \not R c$

$= -8 < 0$

$\therefore R$ is not transitive

Hence R is not an equivalence relation.

7. HW For each of the following relations, determine whether the relation is reflexive, symmetric, antisymmetric or transitive. Which of them are partial orders and equivalence relations?

(a) The relation R on Z where aRb if $a \leq b$

(b) R is a relation on Z where xRy if $x+y$ is odd

(c) R is a relation on Z where xRy if $x-y$ is even

(d) R is a relation on Z where aRb if $a|b$

(e) On the set A of all lines in R^2, define the relation R for two lines l_1, l_2 by $l_1 R l_2$ if $l_1 \perp r$ to l_2.

(f) R is a relation on Z where aRb if $a \neq b$

	Relation	Reflexive	Symmetric	Antisymmetric	Transitive
a)	$\le$ on Z partial ordering relation	$a \le a \ \forall a \in Z$ $\therefore$ Reflexive	If $a \le b$ then $b \not\le a$ $\therefore$ Not symmetric eg:- $1 \le 2$ but $2 \not\le 1$	If $a \le b$ & $b \le a$ then $a = b$ $\therefore$ Antisymmetric	If $a \le b, b \le c$ then $a \le c$ $\therefore$ Transitive
b)	xRy means $x+y$ is odd Neither equivalence nor partial ordering relation	$x + x = 2x$ even $\ne$ odd $x \not R x$ $\therefore$ Not reflexive	If xRy then $x+y$ is odd $y+x$ is odd ie yRx. $\therefore$ symmetric	If xRy and yRx then $x \ne y$ $\therefore$ Not antisymmetric	If xRy, yRz $x+y = \text{odd} \Rightarrow 2m+1$ $y+z = 2n+1$ $x+2y+z = 2(m+n)+2$ $x+z = 2(m+n)+2-2y$ $= 2(m+n-y+1)$ $= \text{even} \ne \text{odd}$ $x \not R z$. $\therefore$ Not transitive.
c)	xRy means $x-y$ is even. Equivalence Relation (Reflexive, symmetric, Transitive)	$x - x = 0$ xRx considering 0 as an even number $\therefore$ reflexive	If xRy, $x-y$ is even. Illly $y-x$ is also even ie yRx $\therefore$ symmetric	If xRy & yRx then $x \ne y$ $\therefore$ Not antisymmetric	If xRy, yRz $x-y = 2m$ $y-z = 2n$ $x-z = 2m+2n$ $x-z = 2(m+n)$ $=$ even xRz $\therefore$ Transitive
d)	aRb means $a\|b$ partial order relation (Reflexive, Antisymmetric, Transitive)	$a\|a$ $\therefore aRa$ Reflexive	If aRb ie $a\|b$. but $b \nmid a$ $\therefore b \not R a$ not symmetric	If aRb & bRa ie $a\|b$ and $b\|a$ then $a=b$ antisymmetric	aRb & bRc ie $a\|b$ & $b\|c$ ie $\frac{b}{a} = m$, $\frac{c}{b} = n$ $\frac{c}{b} \times \frac{b}{a} = nm$ $\frac{c}{a} = k$ ie $a\|c$ $\therefore aRc$ $\therefore$ Transitive

8. Give an example of a relation that is both symmetric and antisymmetric.

Ans :- R be a relation on $A = \{1, 2\}$

$R = \{(1,1), (2,2)\}$

9. Give an example of a relation that is neither symmetric nor antisymmetric

Ans $R = \{(1,3), (3,1), (2,3)\}$ R is a relation on $\{1, 2, 3\}$

★ Definition

Irreflexive Relation.

A relation R on A is called irreflexive if $\forall a \in A$ $(a, a) \notin R$.

10. Give an example of a relation on A that is irreflexive, transitive but not symmetric

Ans R be a relation on $\mathbb{Z}$ defined by

xRy if $x < y$

Partially ordered Set (Poset)

A set together with a partial order relation is called a poset.

ie, The pair (A, R) is called a partially ordered set or poset if R is a partial order relation.

Eg:- $(\mathbb{R}, \leq)$, $(P(A), \subseteq)$, $(\{1,2,3,4\}, /)$ are examples of posets.

(less than or equal to; subset of; divides)

$\mathbb{R}$ - set of real numbers
$P(A)$ - set of all subset of A

Hasse Diagram

It is the pictorial representation of a poset.

Eg:- Consider the poset $(\{1,2,3,4\}, /)$, Draw the hasse diagram.

ie $A = \{1,2,3,4\}$ $\quad xRy$ if x/y

Then $R = \{(1,1), (1,2), (1,3), (1,4), (2,2), (2,4), (3,3), (4,4)\}$

How to draw a Hasse diagram.

① Step 1: Eliminate all reflexive relations.

ie $R = \{(1,2), (1,3), (1,4), (2,4), \quad \}$

② Step 2: Eliminate all transitive relations.

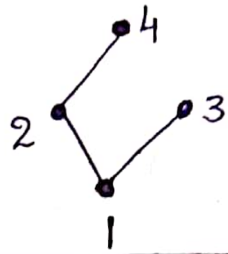
$R = \{(1,2), (2,4), (1,3)\}$

③ Step 3: Draw a line segment from x upto y if xRy.

like this figure

here we can draw a line segment from 1 to 2. 2 to 4 and 1 to 3.

$\therefore$ The hasse diagram is



<u>Note: 1</u> Draw a line segment from x to y if there is nothing "in between" x and y

<u>Note: 2</u>

Read the diagram from bottom to top, and it is not necessary to direct any edges.

2) Draw the Hasse diagram for $(\{1,2,4,8\}, /)$

$A = \{1, 2, 4, 8\}$ $\quad$ xRy means x/y.

Ans:- $R = \{ (1,1), (2,2), (4,4), (8,8),$
$(1,2), (1,4), (1,8), (2,4), (2,8), (4,8)\}$

<u>Step 1</u> :- Eliminate all reflexive relations.

$R = \{ (1,2), (1,4)^*, (1,8)^*, (2,4), (2,8)^*, (4,8)\}$

<u>Step 2</u>:- Eliminate all transitive relations.

~~R={(1,2),~~

$(1,2), (2,4) \Rightarrow (1,4)$ (eliminate)

$(1,2), (2,8) \Rightarrow (1,8)$ (eliminate)

$(2,4), (4,8) \Rightarrow (2,8)$ (eliminate)

<u>Step:3</u> $\therefore$ $R = \{ (1,2), (2,4), (4,8)\}$

Now connect the elements from bottom to top using line segments.

is the hasse diagram

Total order (Linear Order) (Chain)

If (A, R) is a poset, we say that A is totally ordered (or linearly ordered or chain) if for all $x, y \in A$ either xRy or yRx, then R is called a total order or linear order

i.e., every pair of elements are comparable

Eg:- $(N, \leq)$ is a total order relation.

i.e on the set N, the relation R defined by xRy if $x \leq y$ is a total order.

<u>Note</u>:- If we draw the hasse diagram of a totally ordered set, it will be a chain.

Q:2 is a totally ordered set, because its hasse diagram is a chain

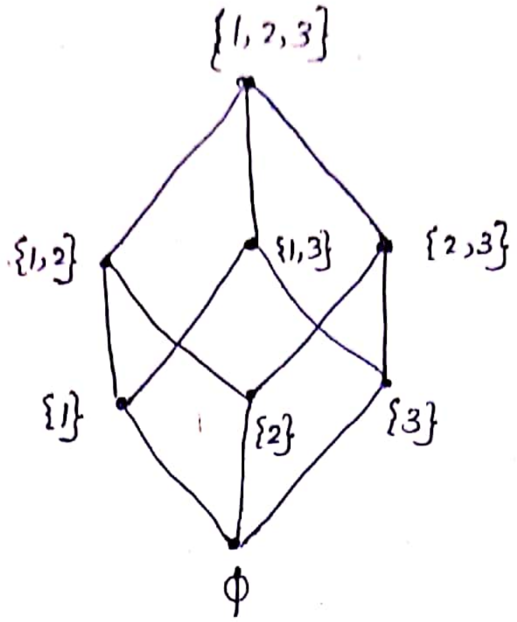
3) Let $U = \{1, 2, 3\}$, $(P(U), \subseteq)$ be a poset.

Draw Hasse diagram.

$P(U) = \{\phi, \{1\}, \{2\}, \{3\}, \{1,2\}, \{1,3\}, \{2,3\}, \{1,2,3\}\}$

$R = \{(\phi, \phi), (\phi, \{1\}), (\{1\}, \{1\}) \ldots \ldots (\{1,2,3\}, \{1,2,3\})$
$(\{1\}, \{1,2\}) (\{1\}, \{1,3\}) (\{1\}, \{1,2,3\}), (\{2\}, \{1,2\}), \{\{2\}, \{2,3\}\} (\{2\}, \{1,2,3\})$
$(\{3\}, \{1,3\}), (\{3\}, \{2,3\}), (\{3\}, \{1,2,3\}), (\{\{1,2\}, \{1,2,3\}) (\{1,3\}, \{1,2,3\})$
$(\{2,3\}, \{1,2,3\})\}$

Hasse Diagram



Maximal and Minimal element.

Let (A, R) be a poset. An element $b \in A$ is called a maximal element if no element of A is larger than (succeeds) b.

An element $a \in A$ is called a minimal element if no other element of A is less than (preceds) a.

ie a is a minimal element, if no edge enters a from below and

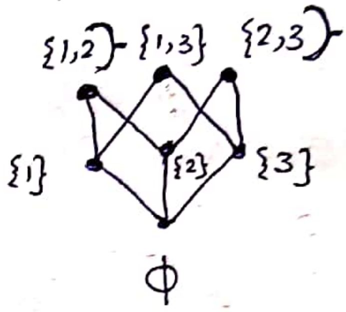
b is a maximal element, if no edge leaves b (in the upward direction).

Eg:- In the above example

(a) $U = \{1, 2, 3\}$ and $A = \mathcal{P}(U)$. Consider the poset $(A, \subseteq)$

$U = \{1, 2, 3\}$ is maximal and ϕ is minimial for $(A, \subseteq)$ (see figure in page no: 32)

(b) Let B be the collection of proper subsets of $U = \{1, 2, 3\}$. Consider $(B, \subseteq)$.



Maximal elements are $\{1,2\}, \{1,3\}, \{2,3\}$

Minimal element is ϕ.

---

1) Find the maximal and minimal elements of the poset $(\mathbb{Z}, \leq)$.

Ans:- The Hasse diagram of $(\mathbb{Z}, \leq)$ is a chain (every pair of elements are related under $\leq$. ∴ Hasse diagram is a chain)

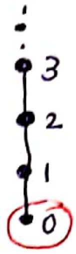
$\mathbb{Z}$: set of integers $\{\ldots -3, -2, -1, 0, 1, 2, 3, \ldots\}$

It is a poset with ~~neight~~ neither a maximal or minimal element.

2) What is the minimal and maximal element of the poset $(N, \leq)$

Ans:- Minimal element is 0

No maximal element



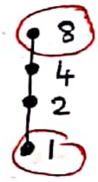
Hasse diagram of $(N, \leq)$

3) Identify the maximal and minimal elements of the following posets. Draw their Hasse diagrams.

(a) $(\{1,2,4,8\}, /)$ (b) $(\{2,3,5,7\}, /)$

(c) $(\{2,3,5,6,7,11,12,35,385\}, /)$

Ans:- (a) Refer pg: no 30 (see the hasse diagram.)

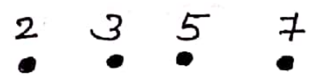


Maximal element : 8 (unique)

Minimal element : 1 (unique)

(b) $A = \{(2,2), (3,3), (5,5), (7,7)\}$

Hasse diagram is

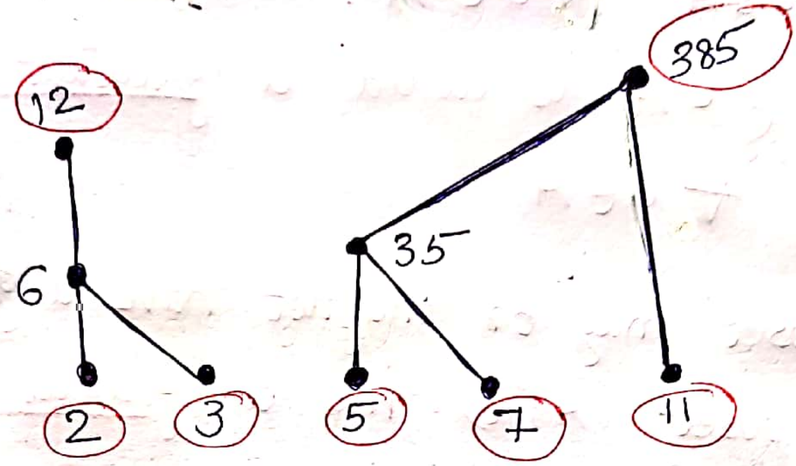


Maximal elements : 2, 3, 5, 7

Minimal element : 2, 3, 5, 7

$\{2,3,5,7\}$ is both maximal & minimal elements.

ⓒ A = { No need to consider reflexive relations,
(2,6), ~~(2,12)~~, (3,6), ~~(3,12)~~, (5, 35), ~~(5, 385)~~,
(6,12), (7,35), ~~(7, 385)~~, (11, 385), (35, 385)}



Maximal elements; 12, 385

Minimal elements; 2, 3, 5, 7, 11

Note :

① A Hasse diagram can have all isolated vertices.

② It can also have two or more connected pieces.

③ If (A,R) is a poset and A is finite, then A has both a maximal and minimal element.

④ A poset can have more than one maximal and minimal elements.

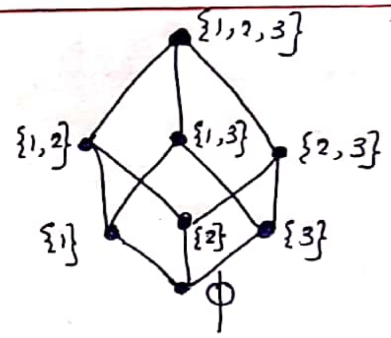
LEAST & GREATEST ELEMENTS OF A POSET

If (A, R) is a poset, then an element $x \in A$ is called a least element if xRa for all $a \in A$ $\forall a \in A$.

Element $y \in A$ is called a greatest element if aRy $\forall a \in A$

Note: A poset may or maynot have a greatest & least element.
If the poset (A, R) has a greatest (least) element, then that element is unique.

Eg: ① $(\mathcal{P}(U), \subseteq)$ $U = \{1,2,3\}$

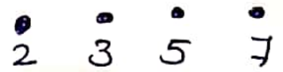


greatest elemet $= \{1,2,3\} = U$
least element $= \phi$

② $(\{1,2,4,8\}, /)$

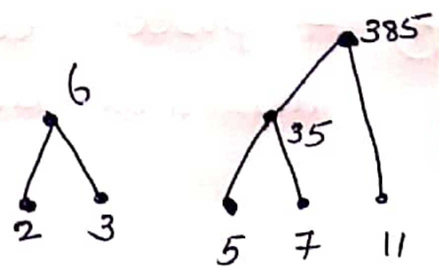
greatest element $= 8$
least element $= 1$

③ $(\{2,3,5,7\}, /)$



no greatest & least element.

④ $(\{2,3,5,6,7,11,12,35,385\}, /)$



no greatest & least element.

Lower bound & upper bound.

greatest lower bound (glb) & least upper bound (lub)

Let (A, R) be a poset with $B \subseteq A$

$x \in A$ is called a lower bound of B if xRb (x, b) $\forall b \in B$

$y \in A$ is called an upper bound of B if bRy (b, y) $\forall b \in B$

Greatest lower bound of a set B (glb)

An element $x' \in A$ is called a glb of B, if

(i) x' is a lower bound of B

(ii) for all other lower bounds x'' of B we have x' must be the greatest $(x'' R x')$ OR (x'', x')

Least upper bound of a set B (lub)

An element $y' \in A$ is called a lub of B, if

(i) y' is an upper bound of B

(ii) for all other upper bounds y'' of B we must have y' as the least $(y' R y'')$ OR (y', y'')

Definition :-

I Consider $\boxed{(P(u), \subseteq)}$, u be any set and $P(u)$ be the set of all subsets of u (power set of u)

important

$\boxed{\begin{array}{l} glb\{A, B\} = A \cap B \\ lub\{A, B\} = A \cup B \end{array}}$ (glb is defined by intersection) (lub is defined by union)

1

II $\boxed{(Z^+, /)}$ Z^+ – set of positive integers
/ – The relation divides.

important
$$\boxed{\begin{array}{l} glb\{a,b\} = gcd\{a,b\} \text{ greatest common divisor} \\ lub\{a,b\} = lcm\{a,b\} \text{ least common multiple} \end{array}}$$

III $\boxed{(R, \leq)}$ R – set of real numbers
$\leq$ – The relation less than or equal to

important
$$\boxed{\begin{array}{l} glb\{a,b\} = min\{a,b\} \\ lub\{a,b\} = max\{a,b\} \end{array}}$$

1) Let $U = \{1,2,3,4\}$. Consider the poset $(P(U), \subseteq)$
If $B = \{\{1\}, \{2\}, \{1,2\}\}$. Find all the upperbounds, least upper bound and greatest lower bound of B.

Ans:- upper bounds of B are $\{\{1,2\}, \{1,2,3\}, \{1,2,3,4\}\}$
least upper bound of B is $\{1,2\}$. $\in B$. $\boxed{\text{using formula } \{1\}\cup\{2\}\cup\{1,2\} = \{1,2\}}$
~~least~~ glb of $B = \{1\}\cap\{2\}\cap\{1,2\}. = \phi \notin B$.

2) Consider $(R, \leq)$ the poset
If $B = [0,1]$ → closed interval. Find glb and lub of B

Ans:- $glb\,[0,1] = min\,[0,1] = 0 \in B$
$lub\,[0,1] = max\,[0,1] = 1 \in B$

3) $(R, \leq)$ be a poset

Let $B = \{ q \in Q \,/\, q^2 < 2 \}$. Find glb & lub

Ans:- $B = \{ (-\sqrt{2}, \sqrt{2}) \in Q \}$

$\therefore$ glb $B = \min(-\sqrt{2}, \sqrt{2}) = -\sqrt{2} \notin B$

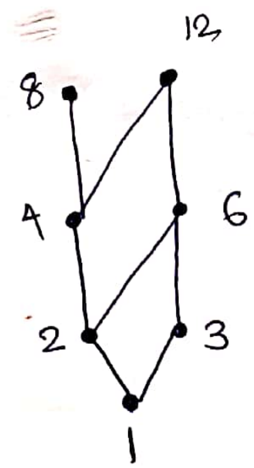
lub $B = \max(-\sqrt{2}, \sqrt{2}) = \sqrt{2} \notin B$

<u>Note</u>:- If $(\overset{\text{set}}{A}, \overset{\text{relation}}{R})$ is a poset and $B \subseteq A$ then B has at most one lub (glb).

<u>Problems</u>

1. Draw the hasse diagram of (A, R) where $A = \{1, 2, 3, 4, 6, 8, 12\}$ and R defined by aRb if a divides b.

Ans:- $R = \{$ Avoid all reflexive relations,
$(1,2), (1,3), (1,4), (1,6), (1,8), (1,12)$
$(2,4), (2,6), (2,8), (2,12)$
$(3,6), (3,12)$
$(4,8), (4,12)$
$(6,12) \}$



(2.) Draw the Hasse diagram for (A, R). Let $A = \{1, 2, 3, 6, 9, 18\}$
H.W define R on A by $x|y$.

(3) Draw the Hasse diagram for the poset $(P(U), \subseteq)$
H.W where $U = \{1, 2, 3, 4\}$

(4.) Let $U = \{1, 2, 3, 4\}$. Let $(P(U), \subseteq)$ be a poset.
H.W For each of the following subsets B of $P(U)$ determine the lub and glb of B.

a) $B = \{\{1\}, \{2\}\}$

b) $B = \{\{1\}, \{2\}, \{3\}, \{1, 2\}\}$

c) $B = \{\phi, \{1\}, \{2\}, \{1, 2\}\}$

d) $B = \{\{1\}, \{1, 2\}, \{1, 3\}, \{1, 2, 3\}\}$

e) $B = \{\{1\}, \{2\}, \{3\}, \{1, 2\}, \{1, 3\}, \{2, 3\}\}$

(5) Define the relation R on $\mathbb{Z}$ by aRb if $a-b$ is a nonnegative even integer. Verify that R defines
H.W a partial order for $\mathbb{Z}$. Is this partial order a total order?

Lattices:

A lattice is a poset (A, R) in which for every pair of elements $a, b \in A$ has a least upper bound (lub) and a greatest lower bound (glb)

We denote lub $\{a,b\}$ by $\boxed{a \vee b}$ OR $\boxed{a+b}$ or $\boxed{a \oplus b}$ call it the join or sum of a and b.

We denote glb $\{a,b\}$ by $\boxed{a \wedge b}$ or $\boxed{a \cdot b}$ or $\boxed{ab}$. call it the meet or product of a and b.

Eg:- 1) The poset $(N, \leq)$ is a lattice with $glb\{x,y\} = min\{x,y\}$, $lub\{x,y\} = max\{x,y\}$ for all $x, y \in N$

2) Let S be any set. The poset $(P(S), \subseteq)$ is a lattice with $glb\{A, B\} = A \cap B$, $lub\{A, B\} = A \cup B$ $\forall A, B \subseteq S$.

3) The poset $(Z^+, /)$ is a lattice with

$$glb\{a,b\} = gcd\{a,b\} \quad \forall a, b \in Z^+$$
$$lub\{a,b\} = lcm\{a,b\}$$

1) Check whether $(D_{35}, /)$ and $(D_{42}, /)$ are lattices. Draw their Hasse diagram.

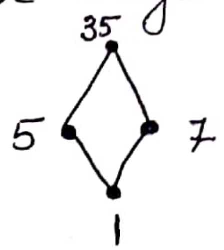
Note:- D_n : The set of all positive divisors of n where n is a +ve integer.

Ans:- $D_{35} = \{1, 5, 7, 35\}$

$(D_{35}, /)$ is a poset.

$R = \{(1,5), (1,7), (1,35), (5,35), (7,35)\}$

Hasse diagram.



glb $\{a,b\}$ = gcd of $\{a,b\}$

lub $\{a,b\}$ = lcm of $\{a,b\}$

pairs	glb $\{a,b\}$ = gcd	lub $\{a,b\}$ = lcm
$\{1,5\}$	1	5
$\{1,7\}$	1	7
$\{1,35\}$	1	35
$\{5,7\}$	1	35
$\{5,35\}$	5	35
$\{7,35\}$	7	35

$\therefore$ every pair of D_{35} has a glb and lub

$\therefore (D_{35}, /)$ is a lattice.

2) $(D_{42}, /)$ is a poset

H.W $D_{42} = \{1, 2, 3, 6, 7, 14, 21, 42\}$

Matrix table for glb and lub.

Eg:- $(D_{35}, /)$ see page: 42.

Matrix table for glb (gcd)

glb(·)	1	5	7	35
1	1	1	1	1
5	1	5	1	5
7	1	1	7	7
35	1	5	7	35

It is a symmetric matrix.

Matrix table for lub (lcm)

lub(+)	1	5	7	35
1	1	5	7	35
5	5	5	35	35
7	7	35	7	35
35	35	35	35	35

It is also a symmetric matrix

(·) – (+)

MEET- JOIN MATRIX TABLE (Together)

(glb) – (lub)

· \ +	1	5	7	35
1	1	5	7	35
5	1	5	35	35
7	1	1	7	35
35	1	5	7	35

→ lub (+)

↓ glb (·)

* The entries of the main diagonal elements are the same as in table of glb or table of lub.

* It is not a symmetric matrix

* The entries below the main diagonal ~~elements~~ can be obtained from the entries below main diagonal elements ~~for~~ of the matrix table for glb.

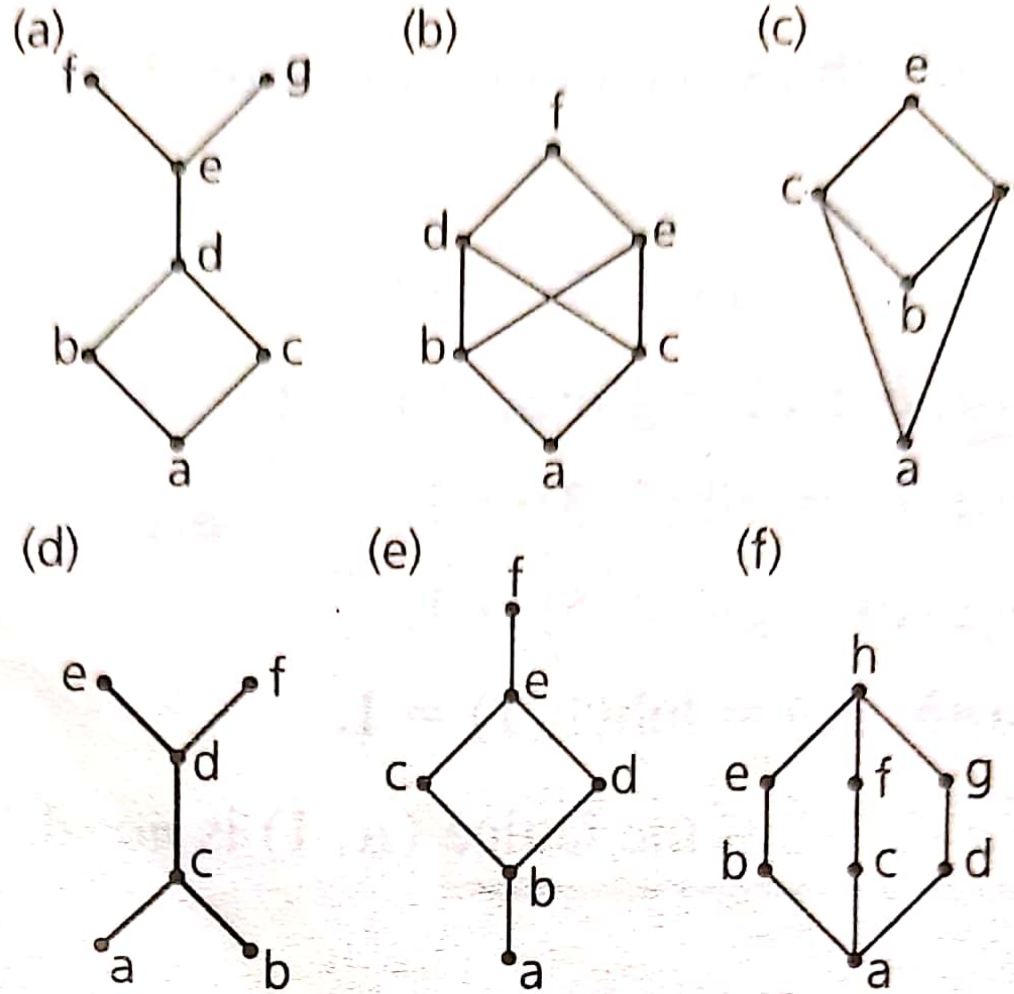
* |||ly the entries above the main diagonal can be obtained from entries above the main diagonal elements ~~for~~ the matrix table for lub

H.W 1) Draw the matrix table for glb and lub of $(D_{42}, /)$. Construct the meet and join table for the same.

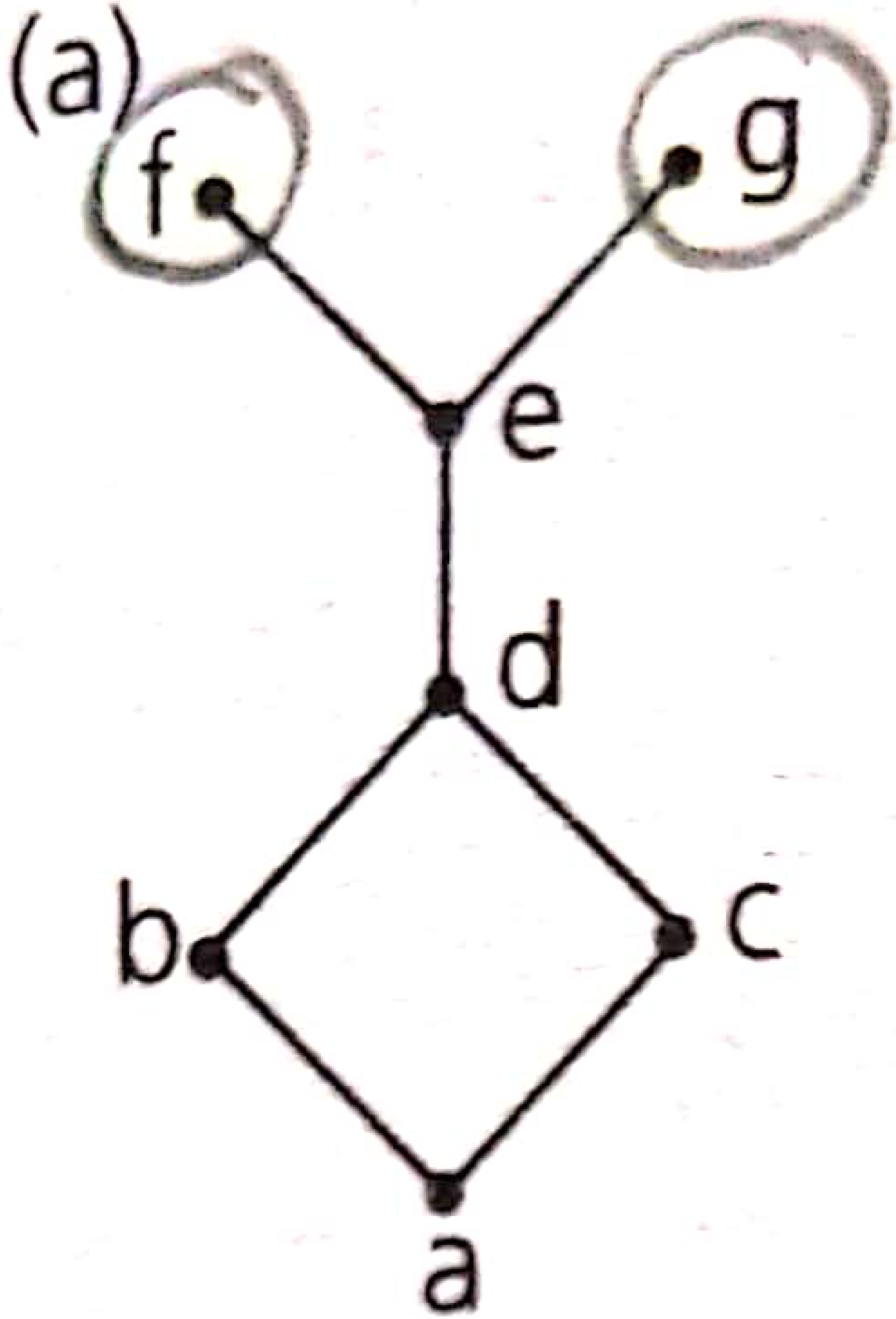
2) H.W For each of the following posets, draw the Hasse diagram and determine whether the poset is a lattice. Construct the table for glb, lub and meet - join table.

ⓐ $(D_8, /)$ ⓑ $(D_6, /)$, ⓒ $(D_{20}, /)$ ⓓ $(D_{24}, /)$, ⓔ $(D_{30}, /)$

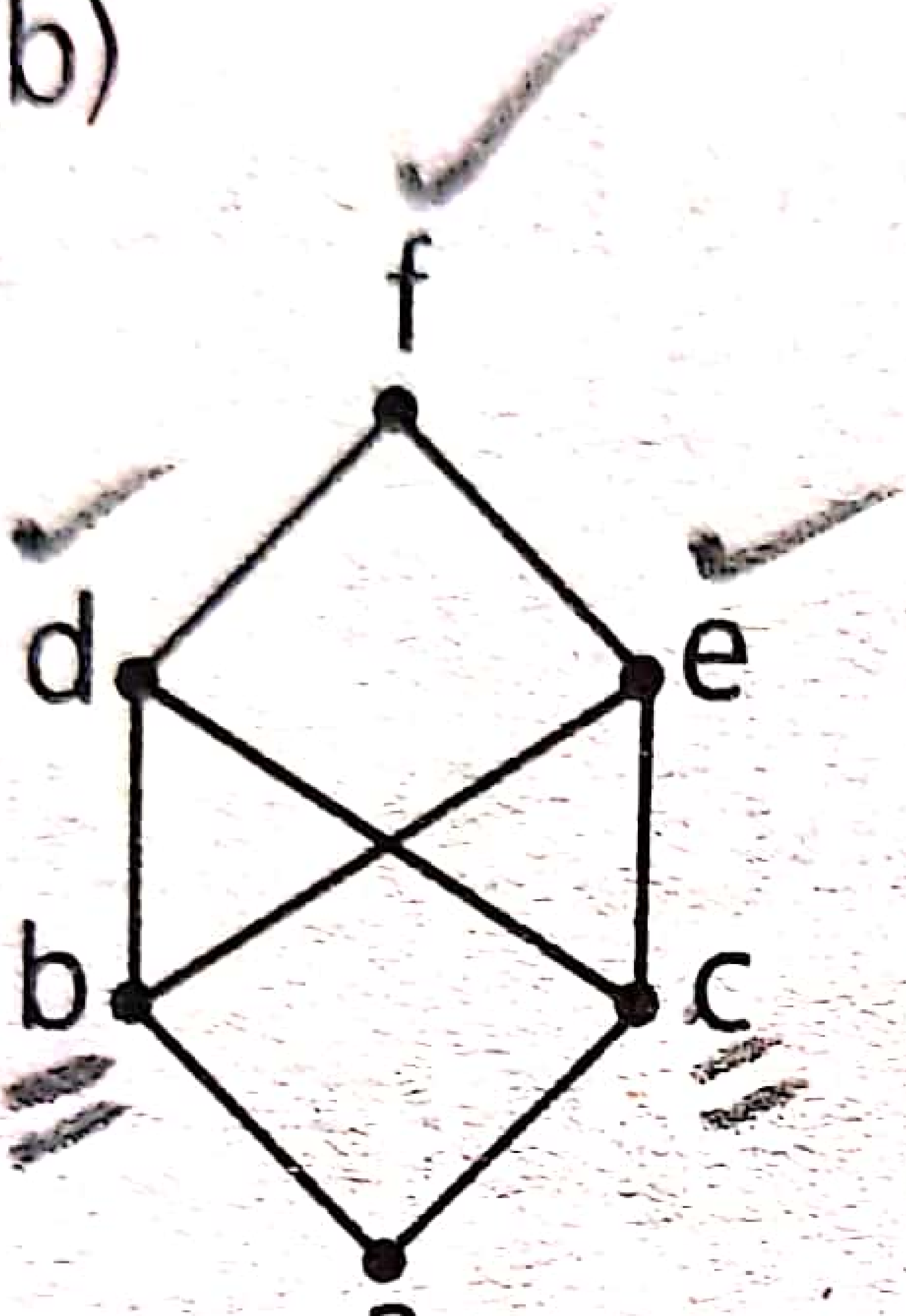
2. State whether the following Hasse diagram represent a lattice or not?



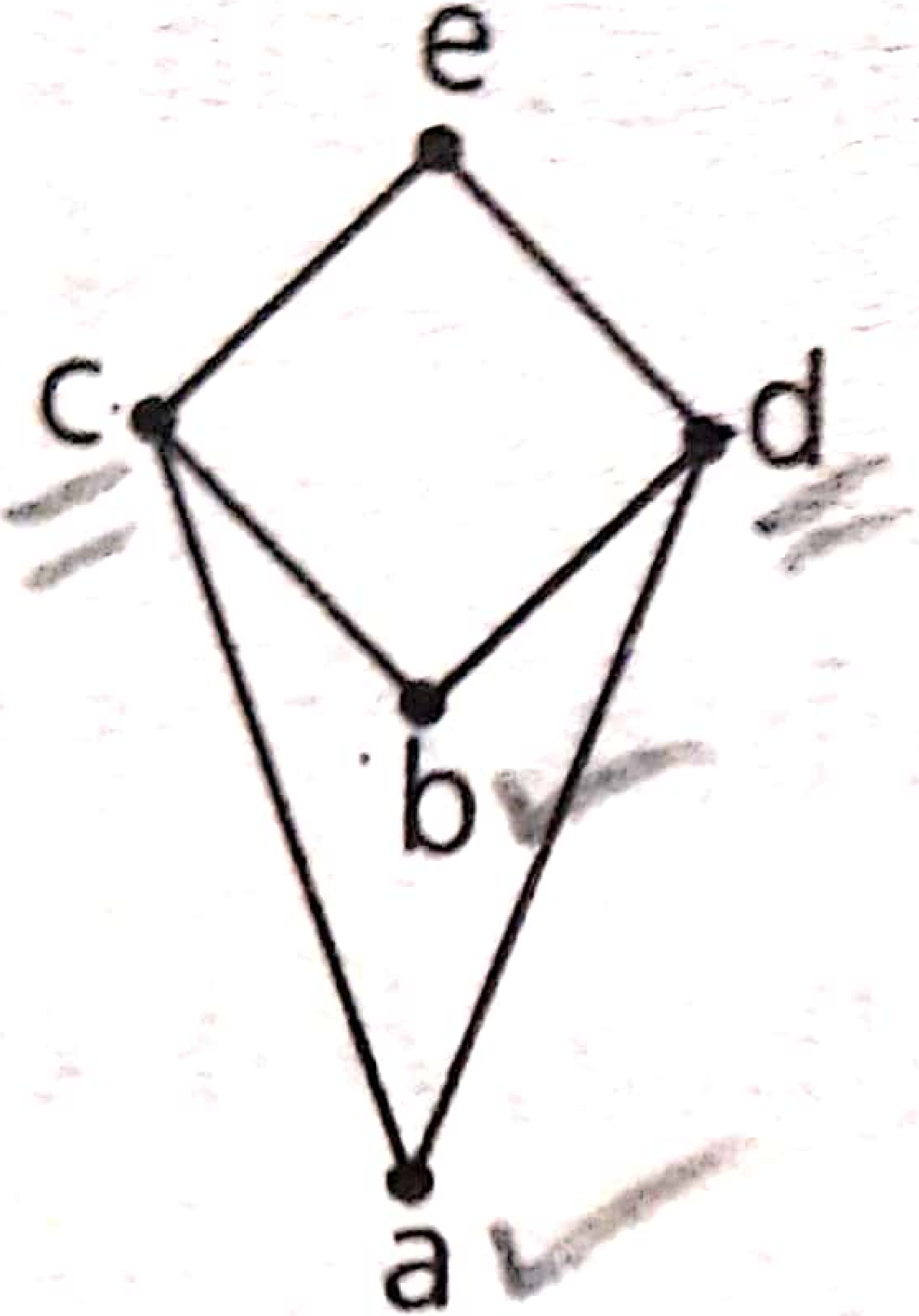
Scanned with CamScanner


(a)
f
g
e
d
b
c
a

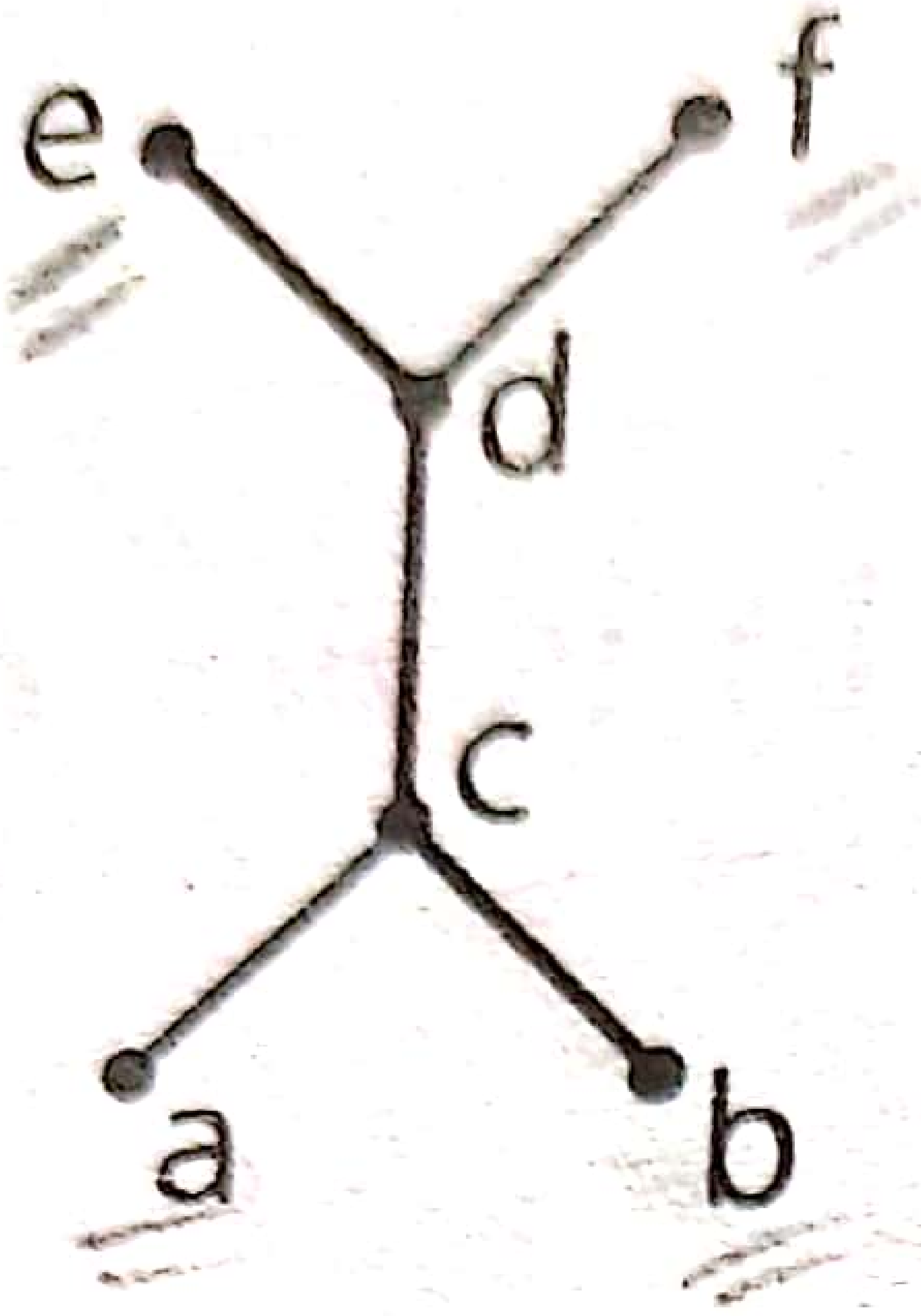
(b)



(c)

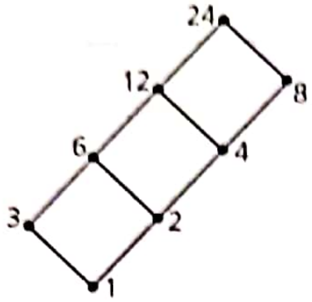


(d)

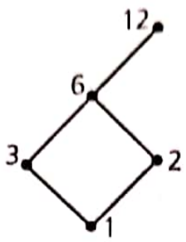


EXAMPLE 7.81 If D_n is the set of all positive divisors of n, then $(D_n, |)$ is a sublattice of the lattice $(Z^+, |)$.

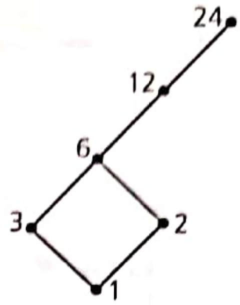
EXAMPLE 7.82 Find all sublattices of D_{24} that contains at least five elements. Give examples of posets which are not sublattices, *with respect to* D_{24}.



D_{24}

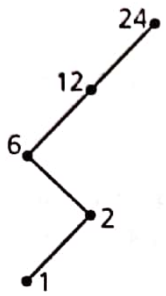


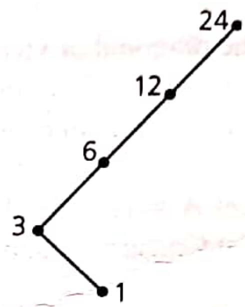
$B_1 = \{1, 2, 3, 6, 12\}$
sublattice



$B_2 = \{1, 2, 3, 6, 12, 24\}$
sublattice

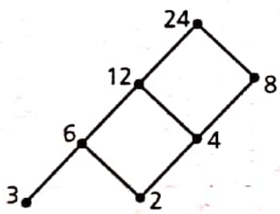
Figure 7.30



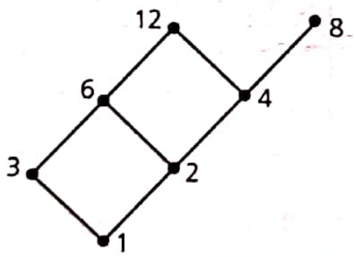


$B_3 = \{1, 2, 6, 12, 24\}$ sublattice. $B_4 = \{1, 3, 6, 12, 24\}$ sublattice.

The following are *not* sublattices



$B_5 = \{2, 3, 4, 6, 8, 12, 24\}$ is not sublattice because glb(3, 2) does not exist.

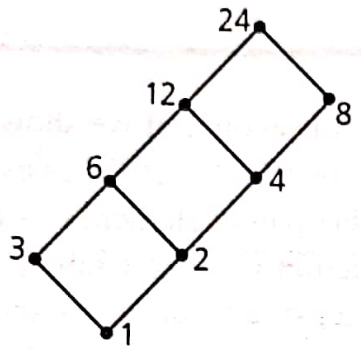


$B_6 = \{1, 2, 3, 4, 6, 8, 12\}$ is not sublattice because lub{8, 12} does not exist.

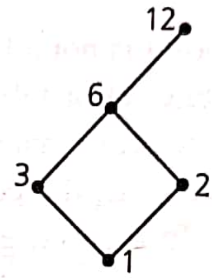
Sublattice: Let (A, R) be a lattice. A nonempty subset B of A is called a sublattice of A for any $a, b \in B$, $a \vee b$ and $a \wedge b \in B$.

If D_n is the set of all positive divisors of n, then $(D_n, |)$ is a sublattice of the lattice $(Z^+, |$

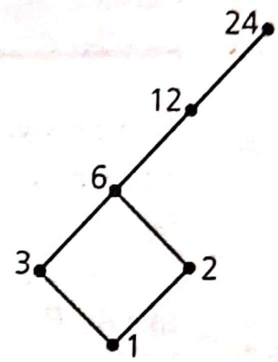
Find all sublattices of D_{24} that contains at least five elements. Give examples of pose which are not sublattices, *with respect to* D_{24}.



D_{24}

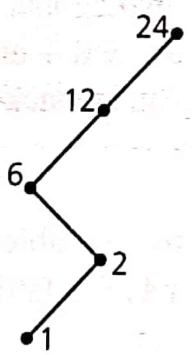


$B_1 = \{1, 2, 3, 6, 12\}$
sublattice

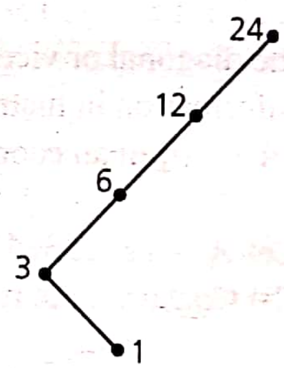


$B_2 = \{1, 2, 3, 6, 12, 24\}$
sublattice

Figure 7.30

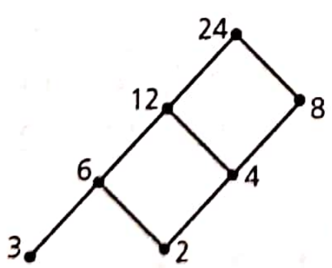


$B_3 = \{1, 2, 6, 12, 24\}$ sublattice.

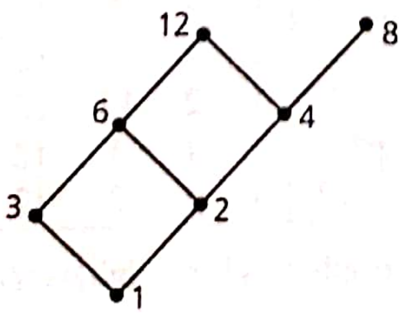


$B_4 = \{1, 3, 6, 12, 24\}$ sublattice.

The following are *not* sublattices



$B_5 = \{2, 3, 4, 6, 8, 12, 24\}$ is not sublattice because glb(3, 2) does not exist.



$B_6 = \{1, 2, 3, 4, 6, 8, 12\}$ is not sublattice because lub{8, 12} does not exist.

Properties of glb and lub:

The following properties follow from the definitions of glb and lub.

1. $\text{glb}(x, y) \leq x$ and $\text{glb}(x, y) \leq y$
   or $x \cdot y \leq x$ and $x \cdot y \leq y$
2. $m \leq x$ and $m \leq y \Rightarrow m \leq \text{glb}(x, y)$
   or $m \leq x$ and $m \leq y \Rightarrow m \leq x \cdot y$
3. $x \leq \text{lub}(x, y)$ and $y \leq \text{lub}(x, y)$
   or $x \leq x + y$ and $y \leq x + y$
4. $x \leq u$ and $y \leq u \Rightarrow \text{lub}(x, y) \leq u$
   or $x \leq u$ and $y \leq u \Rightarrow x + y \leq u$

Properties of Lattices:

Let $(A, R) = (A, \leq) = [A, \cdot, +]$ be a lattice. For any $x, y \in A$, then

I. $\left.\begin{array}{ll} \text{(a)} & x + x = x \\ \text{(b)} & x \cdot x = x \end{array}\right\}$ idempotent

II. $\left.\begin{array}{ll} \text{(a)} & x + y = y + x \\ \text{(b)} & x \cdot y = y \cdot x \end{array}\right\}$ commutative

III. (a) $x + (y + z) = (x + y) + z$ *ie*, $+$ is associative

(b) $x\,(yz) = (xy)z$ *ie*, $\cdot$ is associative

IV. $\left.\begin{array}{ll} \text{(a)} & x + (x \cdot y) = x \\ \text{(b)} & x \cdot (x + y) = x \end{array}\right\}$ Absorption

Some Special Lattices

1) **Complete lattice :**

A lattice is called complete if each of its non-empty subsets have a lub and glb

The greatest element of a lattice if it exist is denoted by I and is known as unit element. Similarly the least element if it exist is denoted by 0 and is known as zero element. Every complete lattice must have I and 0.

eg 1): Every finite lattice is complete $(D_{20}, |)$ is complete

2): Consider the infinite lattice $(Z^+, \leq)$. Now the infinite subset consisting of even positive integers has no lub. So $(Z^+, \leq)$ is not complete

2 **Bounded lattice**

A lattice is said to be bounded if it has a greatest element I and least element 0. The element I and 0 are known as bounds ~~[illegible]~~ of the lattice

Eg 1) : The lattice $(P(A), \subseteq)$ is bounded $(I = A, 0 = \phi)$

2) The lattice $(Z^+, |)$ is not bounded since I doesn't exist (although $0 = 1$)

3) The lattice $(Z, \leq)$ is not bounded since it has neither I or 0

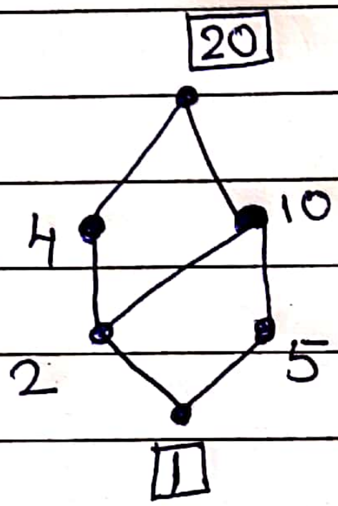
3) **Complemented Lattice**

A lattice is said to be complemented lattice if every element has atleast one complement

DATE:

Eg: 1 $(D_{20}, |) = \{1, 2, 4, 5, 10, 20\}$

$\{(1,2), (1,4), (1,5), (1,10), (1,20),$
$(2,4), (2,10), (2,20),$
$(4,20), (5,10), (5,20), (10,20)\}$



bounds are
$0 = 1$ (least)
$1 = 20$ (greatest)

It is complete.

Eg: 2 $(Z^+, \leq)$ ✗

no lub

no greatest element.

least

Bounds
(least) $0 = 1$
greatest $1 =$ Nil

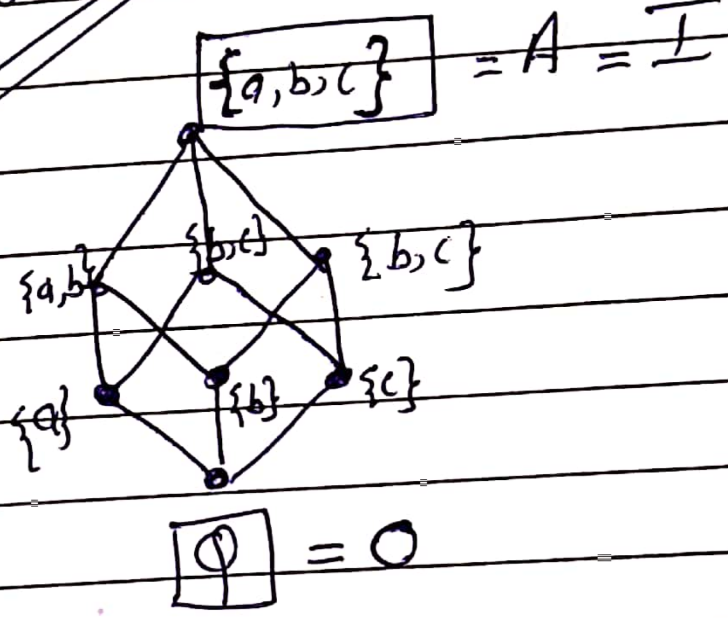
∴ Not complete.

WRITE ON

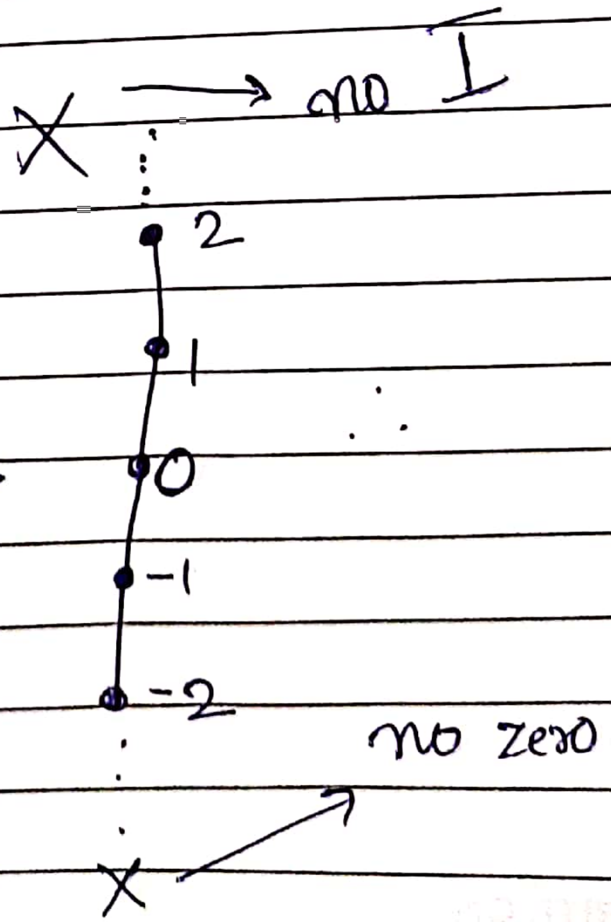
Bounded lattice

$(P(A), \subseteq)$ $A = \{a, b, c\}$

It is bounded



$(Z, \leq)$



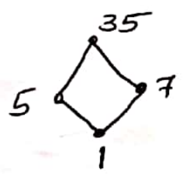
not bounded

* Complement

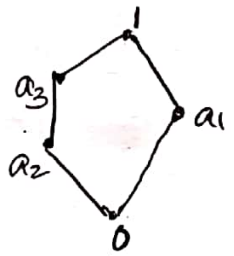
For a bounded lattice an element b is said to be complement of a if $a \cdot b = 0$ and $a + b = I$

* Complement of a is denoted by a'.
* Complement is symmetric (i.e. a is complement of b if b is a complement of a)
* Complement is not unique, need not exist.

Eg: 1) In $B = \{1, 5, 7, 35\}$ complement of 5 is 7 and complement of 35 is 1

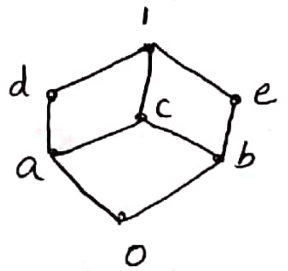


2)



$a_1' = a_2, a_3$
$a_2' = a_1$
$a_3' = a_1$ ∴ (2) complemented lattice
$0' = 1$
$1' = 0$

(3)



Here
$a' = e$
$b' = d$ $(b \vee d = 1 \quad b \wedge d = 0)$
$d' = b, e$
$e' = a, d$.
$c' =$ doesn't exist

∴ not complemented lattice

4) Distributive lattice:

A lattice $[A, \cdot, +]$ is said to be distributive if for any $a, b, c \in A$

1) $a + (b \cdot c) = (a+b) \cdot (a+c)$ (+ is distributed)

$a \cdot (b+c) = (a \cdot b) + (a \cdot c)$ ($\cdot$ is distributed)

eg: $(Z^+, \leq)$ is distributive lattice

A lattice is said to be complemented lattice if every element has at least one complement.

eg:- Let $A = \{1, 2, 3\}$

$(P(A), \subseteq)$ is a complemented lattice

$P(A) = \{ \phi, \{a\}, \{b\}, \{c\}, \{a,b\}, \{b,c\}, \{a,c\}, \{a,b,c\} \}$

elements	complements
ϕ	$\{a,b,c\}$
$\{a\}$	$\{b,c\}$
$\{b\}$	$\{a,c\}$
$\{c\}$	$\{a,b\}$
$\{a,b\}$	$\{c\}$
$\{a,c\}$	$\{b\}$
$\{b,c\}$	$\{a\}$
$\{a,b,c\}$	ϕ

Here.

$\boxed{A = I \quad \phi = 0}$

Note

The complement of any subset B of A is A – B.

WRITE ON

Not complemented

$(D_{20}, /)$

$D_{20} = \{1, 2, 4, 5, 10, 20\}$

greatest
$I = 20$
$O = 1$
least

elements	complements
1	20
2	no complement
4	5
5	4
10	no complement
20	1

$\boxed{\begin{array}{l} gcd = 1 \\ lcm = 20 \end{array}}$

2 & 10 have no complements

∴ Not complemented.

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Distributive lattice

A lattice $[A, \cdot, +]$ is said to be distributive lattice if for any $a, b, c \in A$

$$a+(b\cdot c) = (a+b)\cdot(a+c) \quad (+ \text{ is distributive over } \cdot)$$

& $$a\cdot(b+c) = (a\cdot b)+(a\cdot c) \quad (\cdot \text{ is distributive over } +)$$

Eg:- 1) The lattice $[P(A), \cap\, \cup]$ is distributive for any non empty set A.

Reason Consider $B, C, D \subseteq A$

We can prove that $B \cap (C \cup D) = (B\cap C) \cup (B \cap D)$

$B \cup (C \cap D) = (B \cup C) \cap (B \cup C)$

ie $\cap$ is distributive over $\cup$ and

$\cup$ " " $\cap$.

2) $(Z^+, \leq)$ is a lattice. $\rightarrow$

Recall that
(+) lub = max
(·) glb = min

Let $A = \{1, 2, 3, 5, 30\}$. **(a)** Show that $(A, |)$ is a lattice by constructing a meet-join table. **(b)** Prove that $\cdot$ is not distributive over $+$ in this lattice by identifying elements a, b, c in A for which $a \cdot (b + c) \neq a \cdot b + a \cdot c$. **(c)** Prove that $+$ is not distributive over $\cdot$ by showing that $a + (b \cdot c) \neq (a + b) \cdot (a + c)$ for some elements a, b, c in A.

a) Meet-Join table

Table 7.7

$\cdot \backslash +$	1	2	3	5	30
1	1	2	3	5	30
2	1	2	30	30	30
3	1	1	3	30	30
5	1	1	1	5	30
30	1	2	3	5	30

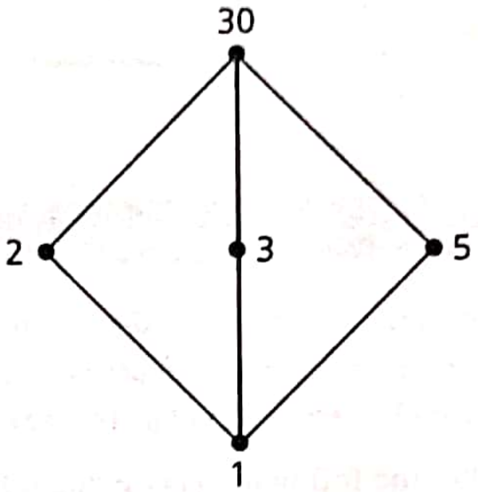


Figure 7.31

The poset $(A, |)$ is a lattice since lub and glb exists for every pair of elements in A^2.

b) Let $a = 2$, $b = 3$, $c = 5$. Then

$$b + c = \text{lub}(b, c) = \text{lub}(3, 5) = 30,$$
$$a \cdot (b + c) = \text{glb}(a, b + c) = \text{glb}(2, 30) = 2,$$
$$a \cdot b = \text{glb}(a, b) = \text{glb}(2, 3) = 1,$$
$$a \cdot c = \text{glb}(a, c) = \text{glb}(2, 5) = 1,$$
$$a \cdot b + a \cdot c = \text{lub}(ab, ac) = \text{lub}(1, 1) = 1.$$

Hence $a \cdot (b + c) = 2 \neq 1 = a \cdot b + a \cdot c$. Thus the lattice $(A, 1)$ is *not* distributive. That is, $\cdot$ is not distributive over $+$.

c) Let $a = 2$, $b = 3$, $c = 5$. Then

$$b \cdot c = \text{glb}(b, c) = \text{glb}(3, 5) = 1,$$
$$a + b \cdot c = \text{lub}(a, bc) = \text{lub}(2, 1) = 2,$$
$$a + b = \text{lub}(a, b) = \text{lub}(2, 3) = 30,$$
$$a + c = \text{lub}(a, c) = \text{lub}(2, 5) = 30,$$
$$(a + b) \cdot (a + c) = \text{glb}(a + b, a + c) = \text{glb}(30, 30) = 1.$$

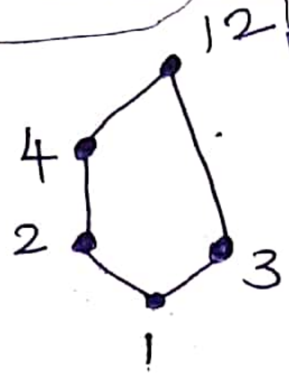
Hence $a + b \cdot c = 2 \neq 1 = (a + b) \cdot (a + c)$. Thus $+$ is not distributive over $\cdot$ in the lattice $(A, |)$. So $(A, |)$ is not distributive.

H.W

1. Find the complement of each element in D_{42}

2. S.T $(A, /)$ where $A = \{1, 2, 3, 4, 12\}$ is not distributive and + is not distributive

Hint :→



prove that

$$a + (b \cdot c) \neq (a+b) \cdot (a+c)$$

take $a = 2$
$b = 3$
$c = 4$

3. Find the complements for the following lattices (A, R)

ⓐ $\{1, 2, 3, 6\}$ ⓑ $\{1, 2, 3, 4, 12\}$ ⓒ $A = \{1, 2, 3, 5, 30\}$

4. Is the poset $(A, /)$ where $A = \{2, 4, 6, 24, 36, 72\}$ a lattice. Explain, Draw the Hasse diagram

PARTITION OF A SET

Let A be a non empty set, a collection of disjoint nonempty subsets of A whose union is A is called a partition of A.

OR

The collection of subsets $A_1, A_2 \cdots A_n$ is a partition of A iff

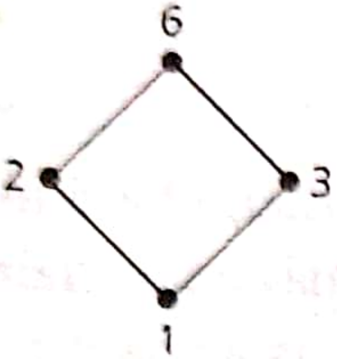
1. $A_i \neq \phi$ for each i (non empty)
2. $A_i \cap A_j = \phi$ for $i \neq j$ (disjoint)
3. $A_1 \cup A_2 \cup \cdots \cup A_n = A$ (union is A).

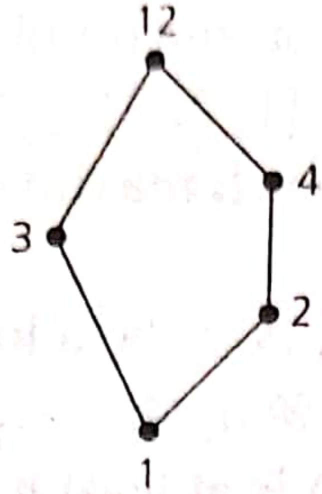
Eg:- $A = \{1,2,3,4,5,6\}$.

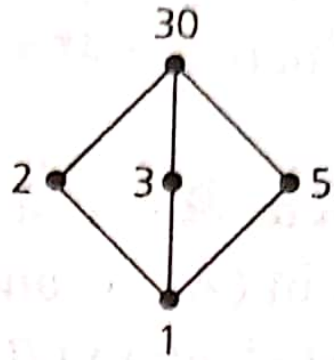
check whether the following are partitions

1. $[\{1,3,5\}, \{2,4\}]$
2. $[\{1,3\}, \{3,5\}, \{2,4,6\}]$
3. $[\{1,2,3\}, \{4,5\}, \{6\}]$

9.







(a) $2' = 3$
$6' = 1$

(b) $3' = 4$ and 2
$1' = 12$

(c) $2' = 3, 5$
$3' = 5, 2$
$5' = 3, 2$
$1' = 30$

Definition 7.20 Given a set A and index set I, let $\emptyset \neq A_i \subseteq A$ for each $i \in I$. Then $\{A_i\}_{i \in I}$ is a *partition* of A if

a) $A = \bigcup_{i \in I} A_i$ and **b)** $A_i \cap A_j = \emptyset$, for all $i, j \in I$ where $i \neq j$.

Each subset A_i is called a *cell* or *block* of the partition.

XAMPLE 7.61 If $A = \{1, 2, 3, \ldots, 10\}$, then each of the following determines a partition of A:

a) $A_1 = \{1, 2, 3, 4, 5\}$, $A_2 = \{6, 7, 8, 9, 10\}$

b) $A_1 = \{1, 2, 3\}$, $A_2 = \{4, 6, 7, 9\}$, $A_3 = \{5, 8, 10\}$

c) $A_i = \{i, i + 5\}$, $1 \leq i \leq 5$.

In these three examples we note how each element of A belongs to *exactly one* cell in each partition.

1. Determine whether each of the following collections of sets is a partition for the given set A. If the collection is not a partition, explain why it fails to be.

a) $A = \{1, 2, 3, 4, 5, 6, 7, 8\}$; $A_1 = \{4, 5, 6\}$, $A_2 = \{1, 8\}$, $A_3 = \{2, 3, 7\}$.

b) $A = \{a, b, c, d, e, f, g, h\}$; $A_1 = \{d, e\}$, $A_2 = \{a, c, d\}$, $A_3 = \{f, h\}$, $A_4 = \{b, g\}$.

Let $\mathcal{R}$ be an equivalence relation on a set A. For each $x \in A$, the *equivalence class of* x, denoted $[x]$, is defined by $[x] = \{y \in A \mid y \mathcal{R} x\}$.

Q) If R is the equivalence relation on the set $A = \{1, 2, 3, 4, 5, 6\}$ given below. Find the partition of A induced by R.

$R = \{(1,1), (1,2), (2,1), (2,2), (3,3), (4,4), (4,5), (5,4), (5,5), (6,6)\}$

$[1] = \{1, 2\}$ $[2] = \{1, 2\}$, $[3] = \{3\}$, $[4] = \{4, 5\}$

$[5] = \{4, 5\}$, $[6] = \{6\}$

$\therefore$ $\{1, 2\}, \{3\}, \{4, 5\}, \{6\}$ is the partition induced by R.

Q). If R is the equivalence relation on Z defined by aRb if $a^2 = b^2$ or $a = \pm b$ find the partition of Z.

Ans:- $Z = \{\ldots -3, -2, -1, 0, 1, 2, 3, \ldots\}$

$R = \{(1,1), (1,-1), (2,2), (2,-2), (0,0), (3,-3), (3,3), \ldots\}$

$[0] = \{0\}$

$[1] = \{1,-1\}$

$[2] = \{2,-2\}$

$[3] = \{3,-3\}$

$\therefore$ Partitions are $[0], [1], [2], [3] \ldots$

or $\{0\}, \{1,-1\}, \{2,-2\} \ldots$

Q. If R is the equivalence relation on the set $A = \{1, 2, 3, 4, 5, 6, 7\}$ defined by aRb if a−b is a multiple of 3. Find the partition of A induced by R.

Ans:- The possible remainders are $\{0, 1, 2\}$

$R = \{(1,1), (1,4), (1,7), (2,2), (2,5), (3,3), (3,6), (4,1), (4,4), (4,7), (5,5), (5,2), (6,3), (6,6), (7,7)\}$

$[1] = \{1, 4, 7\}$

$[2] = \{2, 5\}$.

$[3] = \{3, 6\}$

∴ partition is $\{1,4,7\}, \{2,5\}, \{3,6\}$

OR $[1], [2], [3]$

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3. If $A = \{1, 2, 3, 4, 5\}$ and $\mathcal{R}$ is the equivalence relation on A that induces the partition $A = \{1, 2\} \cup \{3, 4\} \cup \{5\}$, what is $\mathcal{R}$?

4. For $A = \{1, 2, 3, 4, 5, 6\}$, $\mathcal{R} = \{(1, 1), (1, 2), (2, 1), (2, 2), (3, 3), (4, 4), (4, 5), (5, 4), (5, 5), (6, 6)\}$ is an equivalence relation on A. **(a)** What are $[1]$, $[2]$, and $[3]$ under this equivalence relation? **(b)** What partition of A does $\mathcal{R}$ induce?

Q1) If R is the equivalence relation on the set $A = \{1, 2, 3, 4, 5, 6, 7\}$ defined by aRb if $a-b$ is a multiple of 3. Find the partition of A induced by R.

Ans:- $R = \{(1,1), (1,4), (1,7),$
$(2,2), (2,5), (3,3), (3,6),$
$(4,1), (4,4), (4,7), (5,5), (5,2),$
$(6,6), (6,3), (7,7), (7,4), (7,1)\}$

$[4] = [7] = [1] = \{1, 4, 7\}$ (These are numbers in A which give remainder 1 when divided by 3)

$[5] = [2] = \{2, 5\}$ (numbers which give remainder 2 when divided by 3)

$[6] = [3] = \{3, 6\}$ (Elements with remainder 0)

$\therefore$ Partitions are $\{1,4,7\}$, $\{2,5\}$, $\{3,6\}$

Equivalence OR classes: $[1]$, $[2]$, $[3]$.

Note

Congruent Relation ($\equiv$)

$a \equiv b \pmod{n}$ means $a-b$ is a multiple of n.

OR $n \mid (a-b)$.

for the above example, $a \equiv b \pmod 3$ means $a-b$ is a multiple of 3.

if, $a \equiv b \pmod{n}$ then $b \equiv a \pmod{n}$

Q) Define R on $A = \{1,2,3,4,5,6,7,8,9,10,11,12\}$ by $(x,y) \in R$ if $x-y$ is a multiple of 5. (a) Show that R is an equivalence relation on A. (b) Determine the equivalence classes and partitions of A induced by R.

Ans:- (a) Reflexive : $\dfrac{aRa}{a-a=0}$ ($0 = 5 \times 0$ (a multiple of 5))

Symmetric : If $a-b$ is a multiple of 5 then $b-a$ is also a multiple of 5

ie $a-b = 5n$

$b-a = -5n$

$\therefore$ If aRb then bRa.

Transitive: If aRb & bRC

ie $a-b = 5n$.

$b-C = 5m$

$(a-b)+(b-c) = 5n-5m$

$a-c = 5(n-m)$

$a-c$ is a multiple of 5

$\therefore$ aRc

ie aRb & $bRC \Rightarrow \underline{aRc}$

ie R is Reflexive, symmetric & Transitive

$\therefore$ R is an equivalence relation.

(b) $R = \{(1,1), (1,6), (1,11),$
$(2,2), (2,7), (2,12),$
$(3,3), (3,8),$
$(4,4), (4,9)$
$(5,5), (5,10)$
$(6,6), (6,11), (6,1)$
$(7,7), (7,12),$
$(8,8), (8,3),$
$(9,9), (9,4)$
$(10,10), (10,5)$
$(11,11), (11,1),$
$(12,12), (12,7)$
$(12,2)\}$

Easy method:-

Step 1 Identify the possible remainders when divided by 5
$\{0,1,2,3,4\}$

Step 2:-
There are 5 classes. (blocks)

Step 3
* Elements which leave remainder 0
* Elements which leave remainder 1

⋮

equivalence classes

$[1] = \{1,6,11\}$ (with remainder 1)
$[2] = \{2,7,12\}$ (with remainder 2)
$[3] = \{3,8\}$ (with remainder 3)
$[4] = \{4,9\}$ (with remainder 4)
$[5] = \{5,10\}$ (with remainder 0)

Here $[1] = [6] = [11]$
$[2] = [7] = [12]$
$[3] = [8]$
$[4] = [9]$
$[5] = [10]$

$\therefore$ partitions are $\{1,6,11\}, \{2,7,12\}, \{3,8\}, \{4,9\}, \{5,10\}$

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3 Q. If a relation R on the set of integers Z is defined by aRb if $a \equiv b \pmod 4$
Find the partition induced by R.

Ans:- The possible remainders are $\{0,1,2,3\}$

$[0] = \{ \ldots -12, -8, -4, 0, 4, 8, 12 \ldots \}$

$[1] = \{ \ldots -11, -7, -3, 1, 5, 9, 13 \ldots \}$

$[2] = \{ \ldots -10, -6, -2, 2, 6, 10, 14, \ldots \}$

$[3] = \{ \ldots -9, -5, -1, 3, 7, 11, 15, \ldots \}$

∴ ~~The partitions~~

∴ $[0], [1], [2], [3]$ form a partition of R.

4 Q. Let $A = \{1,2,3,4,5\} \times \{1,2,3,4,5\}$ define R on A by $(x_1, y_1) R (x_2, y_2)$ if $x_1 + y_1 = x_2 + y_2$

(a) Verify that R is an equivalence relation on A

(b) Determine the equivalence classes and partition of A induced by R.

Ans:- $A \times A = \{ (1,1), (1,2), (1,3), (1,4), (1,5),$
$(2,1), (2,2), (2,3), (2,4), (2,5)$
$(3,1), (3,2), (3,3), (3,4), (3,5)$
$(4,1), (4,2), (4,3), (4,4), (4,5)$
$(5,1), (5,2), (5,3), (5,4), (5,5) \}$

25 elements

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Reflexive $(x_1, y_1) R (x_1, y_1)$

$$x_1 + y_1 = x_1 + y_2$$

Symmetric If $(x_1, y_1) R (x_2, y_2)$

ie $x_1 + y_1 = x_2 + y_2$

$$x_2 + y_2 = x_1 + y_1$$

then $(x_2, y_2) R (x_1, y_1)$

Transitive : $(x_1, y_1) R (x_2, y_2)$ & $(x_2, y_2) R (x_3, y_3)$

ie $x_1 + y_1 = x_2 + y_2$ & $x_2 + y_2 = x_3 + y_3$

then $x_1 + y_1 = x_3 + y_3$

ie $(x_1, y_1) R (x_3, y_3)$

equivalence classes

$[(1,1)] = \{(1,1)\}$

$[(1,2)] = \{(1,2), (2,1)\}$

$[(1,3)] = \{(1,3), (3,1), (2,2)\}$

$[(1,4)] = \{(1,4), (4,1), (3,2), (2,3)\}$

$[(1,5)] = \{(1,5), (5,1), (3,3), (4,2), (2,4)\}$

$[(2,5)] = \{(2,5), (5,2), (4,3), (3,4)\}$

$[(3,5)] = \{(3,5), (5,3), (4,4)\}$

$[(4,5)] = \{(5,4), (4,5),\}$

$[(5,5)] = \{(5,5)\}$

partitions.

* union will give 25 elements.

* no intersection